Equation systems

Table of contents

Definitions and methods for solving systems

An equation is a statement saying that two expressions are equal, and solving an equation is the process of finding the implied numerical value – or the algebraic expression – equal to a specific variable or expression within the equation.

For example: the volume of water in a pool formed as a cuboid (rectangular box) equals the width of the pool times its length times its depth: $V = wld$. If we know that the volume equals 1 200 m^3 and that its width and length equals 20 m and 30 m (respectively), we can find out its depth by asking the question “for what depth d is it true that $20 \times 30 \times d = 1\,200\ \text{m}^3$?”.

We realize that there only is one d satisfying this equation; a too small depth would give a too small volume and a too large depth would give a too large volume.

In this case, we can very easily obtain the current depth d by solving the equation and inserting the numbers:

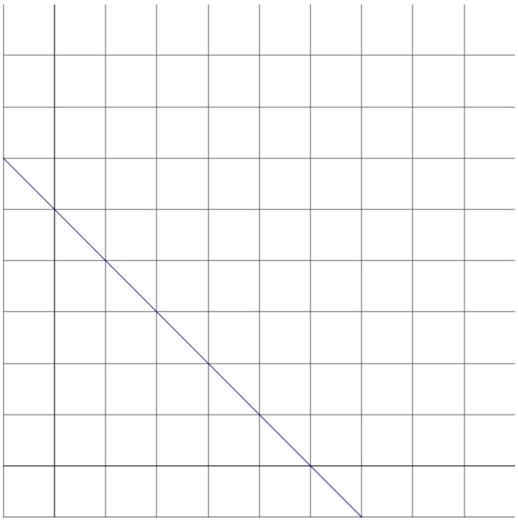
$$V = wld$$

$$d = \frac{V}{wl} = \frac{1200\,\text{m}^3}{20\,\text{m} \times 30\,\text{m}} = 2\,\text{m}$$

However, if we have an equation including *two* (or more) unknown variables, there is not always a single root, because the two unknown variables may compensate for each other. For instance, the equation $x + y = 5$ has infinitely many solutions, i.e. all roots (x, y) satisfying $y = 5 - x$. In the table below, all pairs of numbers exemplify valid roots.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
y	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-11	-12	-13	-14	-15

The function may also be plotted on a plane, where all points on the graph represent valid roots.



Nevertheless, if we have *two* equations including the same unknown variables, we can find the root(s) shared by both equations.

Let us exemplify by solving the following equations:

$$y = 2x \qquad (1)$$

$$y = x + 2 \qquad (2)$$

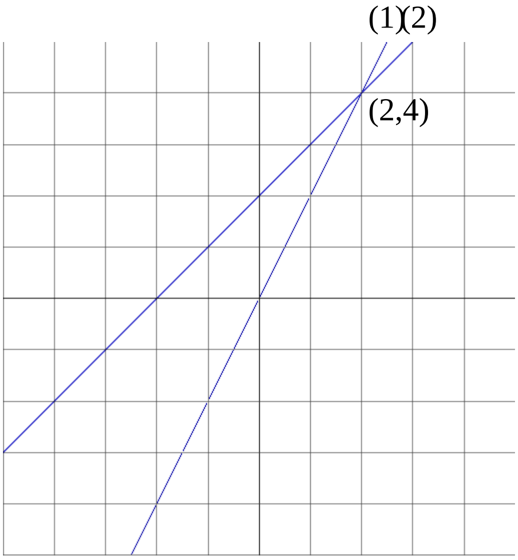
We can create tables and graphs:

(1)

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
y	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40

(2)

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
y	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22



Alone, each equation has infinitely many solutions, but they only have one root in common, $\begin{cases} x = 2 \\ y = 4 \end{cases}$, which consequently is the solution to the *equation system* $\begin{cases} y = 2x \\ y = x + 2 \end{cases}$. Thus, an equation system is a set of equations, and solving an equation system is the process of finding the root(s) satisfying *all* equations in the system.

Algebraic solutions

Principally, there are two simple methods for solving simple equation systems.

By substitution

Assume that we want to solve the following system:

$$\begin{cases} y + 5 = 2x & \qquad (1) \\ 4y = x + 2 & \qquad (2) \end{cases}$$

Using the two equations above, we are able to create a new (implied) equation containing only one of the two unknown variables. This may be accomplished by solving either of the equations with respect to one of the unknowns and insert the expression of this unknown (as a function of the other unknown variable) into the other equation.

$$\begin{cases} y+5=2x & (1) \\ 4y=x+2 & (2) \end{cases}$$

Equation (1) is solved with respect to *y*:

$$\begin{aligned} &y+5=2x && (1) \\ &y=2x-5 \end{aligned}$$

Knowing *y* as a function of *x*, we can substitute *y* with the expression containing *x* at any place. We will do this in equation (2).

$$\begin{aligned} &4y=x+2 && (2) \\ &(1)\to(2) \\ &4(2x-5)=x+2 \end{aligned}$$

Thus, we have obtained a new equation containing only one unknown.

$$\begin{aligned} &4(2x-5)=x+2 \\ &8x-20=x+2 \\ &7x-20=2 \\ &7x=22 \\ &x=\frac{22}{7} \end{aligned}$$

Knowing one of the two variables in the system, there is only one unknown left. Therefore, we substitute *x* in either of the equations with the numerical value of it obtained from equation (2).

$$\begin{aligned} &y=2x-5 && (1) \\ &(2)\to(1) \\ &y=2\left(\frac{22}{7}\right)-5 \\ &y=\frac{44}{7}-5 \\ &y=\frac{9}{7} \end{aligned}$$

Now, we have completely solved the system.

$$\begin{cases} x=\dfrac{22}{7} \\ y=\dfrac{9}{7} \end{cases}$$

By addition

An equation system may also be solved by adding the two equations, forming a new equation with one of the variables eliminated. Let us study an example:

$$\begin{cases} y+2=2x & (1) \\ 4y-5=x & (2) \end{cases}$$

These equations may be added to each other, side by side, forming a new equation. This is meaningful because the new equation will be an implication of the old ones. For instance, if we start with equation (2) and add its left side with the left side of equation (1) and its right side with the right side of equation (1), then we have performed the very same operation on both sides of (2) because the left side of equation (1) – of course – equals the right side of equation (1).

Moreover, the new equation may have one variable term eliminated (removed) if the sum of the variable's previous coefficient equals 0. To make sure that the sum of either of the variable coefficients in the resulting equation will equal 0, we can multiply either of the equations with a real number before performing the addition. In the end, we will obtain a resulting equation with only one unknown, which easily can be solved.

In this case, we see that the *y* term in equation (1) has the coefficient 1, and that the corresponding term in equation (2) has the coefficient 4. If we multiply equation (1) with -4, the resulting equation will get a *y* term having the coefficient $(-4)+4=0$; in other words, the *y* term will be eliminated during addition.

$$(1)\times(-4)$$

$$\begin{cases} -4y-8=-8x & (1) \\ 4y-5=x & (2) \end{cases}$$

We are now able to perform the addition (and elimination):

$$\begin{array}{l} \begin{cases} -4y-8=-8x & (1) \\ 4y-5=x & (2) \end{cases} \\ \hline 0y-13=-7x \end{array}$$

The resulting equation can be simplified to look like

$$-13=-7x\,.$$

Containing only one unknown, we can easily solve this equation:

$$-13=-7x$$

$$x=\frac{-13}{-7}$$

$$x=\frac{13}{7} \qquad (3)$$

To finish solving the system, we must also solve it for *y*.

$$4y - 5 = x \qquad (2)$$

$$(3) \to (2)$$

$$4y - 5 = \frac{13}{7}$$

$$4y = \frac{13}{7} + 5$$

$$4y = \frac{48}{7}$$

$$y = \frac{48}{7 \times 4}$$

$$y = \frac{12}{7}$$

Thus, the equation system has a single root $\begin{cases} x = \frac{13}{7} \\ y = \frac{12}{7} \end{cases}$.

Systems lacking roots

Not all systems can be solved, because the equations not always have any shared roots. For instance, consider the following system:

$$\begin{cases} x + y = 5 \\ x + y = 6 \end{cases}$$

There are no complex numbers *x* and *y* such that their sum equals both five and six at the same time. Thus, if we plot these functions in a coordinate system, we will see two parallel lines having no intersection. Performed on systems lacking roots, the method of substituting will cause contradictive (false) results, in this case $5 = 6$. The method of addition may (here) give the equally false equation $0 = -1$.

Systems having infinitely many roots

If the equations in a system are equivalent to each other, the system will have the same set of solutions as the individual equations themselves. In other words, the second equation does not "narrow" the set of roots from the first equation, or vice versa. Consider the following system as an example.

$$\begin{cases} x + y = 100 & (1) \\ 2x + 2y = 200 & (2) \end{cases}$$

Simplification of equation (2) gives (1). In a coordinate system, these graphs will lay over each other. The system has as many roots as each separate equation has, i.e. infinitely many. In such cases, the method of substitution will give "tautological" results, here $100 = 100$. The same applies for the method of addition ($0 = 0$).

Practical example

A supermarket has bought several packages of sodium chloride (table salt) from the manufacturer. The packages are of two sizes with distinct masses (including the package): 50 g and 100 g. Since the demand for the smaller package is 1.6 as high as the demand for the larger, the supermarket has ordered 1.6 times as many small packages as large packages. At the arrival, the total mass of the packages was measured to 45 000 g. How many large and small packages were delivered to the supermarket?

Solution:

Let n_1 be the number of packages with the mass 50 g and n_2 be the number of packages with the mass 100 g. Then the total mass can be written $n_1 \times 50 + n_2 \times 100 = 45000$ – an equation having infinitely many solutions. The relationship between the numbers of packages of each size can be written $n_1 = 1.6 \times n_2$, which also has infinitely many solutions. Together, however, these equations form a solvable equation system:

$$\begin{cases} n_1 \times 50 + n_2 \times 100 = 45000 & (1) \\ n_1 = 1.6 \times n_2 & (2) \end{cases}$$

The function of n_2 in equation (2) substitutes n_1 in equation (1):

$$\begin{aligned} &n_1 \times 50 + n_2 \times 100 = 45000 \qquad (1) \\ &(2) \to (1) \\ &(1.6 \times n_2) \times 50 + n_2 \times 100 = 45000 \end{aligned}$$

Now we can easily solve (1), having only one unknown:

$$\begin{aligned} &(1.6 \times n_2) \times 50 + n_2 \times 100 = 45000 \\ &80n_2 + 100n_2 = 45000 \\ &180n_2 = 45000 \\ &n_2 = 250 \end{aligned}$$

The numeric value of n_2 then substitutes the variable in equation (2), forming a new equation with only one unknown:

$$\begin{aligned} &n_1 = 1.6 \times n_2 \qquad (2) \\ &(1) \to (2) \\ &n_1 = 1{,}6 \times 250 \\ &n_1 = 400 \end{aligned}$$

Thus, we have found the only possible root to the system:

$$\begin{cases} n_1 = 400 \\ n_2 = 250 \end{cases}$$

Answer: 400 packages with the mass 50 g and 250 packages with the mass 100 g were delivered.