ANDREAS REJBRAND NV3ANV
http://www.rejbrand.se

2006-02-14

Mathematics

The Language of Mathematics

Table of contents

Introduction

Mathematical discussions are always to be stringent. In other words, there should never be any doubt about what is meant, and all conclusions are to be completely certain. In order to succeed in describing objects and events in an exact way, we may use the well-defined language of mathematics. In this document, we shall study the concepts of *sets* and *logic*.

Sets

In many situations, it may be meaningful to study objects by grouping them into "collections". Common examples include a class at school, the inhabitants of a city, and the fruits in a supermarket. These "collections" are called *sets*, and their objects are called *elements*.

Expressing a set

Very simply, a set may be expressed by listing all its elements within brackets, using commas as separators. For instance, if we assume that the set *A* contains the elements *a*, *b* and *c*, we write

$$A = \{a, b, c\}.$$

Example 1

$$\text{days} = \{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}, \text{Sunday}\}$$

$$\text{planets} = \{\text{Mercury}, \text{Venus}, \text{Earth}, \text{Mars}, \text{Jupiter}, \text{Saturn}, \text{Uranus}, \text{Neptune}, \text{Pluto}\}$$

$$\text{isotopes of hydrogen} = \{{}^{1}_{1}\text{H}, {}^{2}_{1}\text{H}, {}^{3}_{1}\text{H}\}$$

There is one set containing no elements at all. This is called the *empty set* and is designated using the symbol $\varnothing$.

Example 2

The set of planets in the universe, besides earth, that the human year 2005 knew to contain living organisms equals $\varnothing$.

Two sets are said to be equal if and only if they contain the very same elements. The order in which the elements are declared does not matter, and nor do reappearances of elements. For instance, if

$$A = \{a, b, c\}$$

and

$$B = \{c, b, a, a\}$$

then $A = B$.

The number of elements existing in a set *A* is written $|A|$. For instance, $|A| = |B| = 3$.

Symbols

The fact that an element *a* is a member of the set *A* is written

$a \in A$ (“*a* is a member of *A*”).

The fact that an element *a* is *not* a member of the set *A* is written

$a \notin A$ (“*a* is not a member of *A*”).

Example 3

We know that

$$\begin{aligned}&\text{Monday} \in \text{days},\\&\text{Earth} \in \text{planets and}\\&{}^{2}_{1}\text{H} \in \text{isotopes of hydrogen}.\end{aligned}$$

We also know that

$$\begin{aligned}&\text{January} \notin \text{days},\\&\text{ISS} \notin \text{planets and}\\&{}^{4}_{2}\text{He} \notin \text{isotopes of hydrogen}\end{aligned}$$

Example 4

The set of numbers *x* satisfying the equation

$$x^2 = 4$$

equals $\{-2,2\}$.

Thus $x \in \{-2,2\}$.

Number systems

A very common set is the set of *the natural numbers*, designated *N*. As there are infinitely many natural numbers, we must abbreviate the set description.

$$N = \{0,1,2,3,4,5,...\}$$

Another common set is the one containing all *integers*, *Z*.

$$Z = \{0,1,-1,2,-2,3,-3,4,-4,5,-5...\}$$

Sometimes the symbol Z^+ is used for the set of all *positive integers* and Z^- for the set of all *negative integers*.

$$Z^{+} = \{1,2,3,4,5...\}$$
$$Z^{-} = \{-1,-2,-3,-4,-5...\}$$

The sets *N*, *Z*, Z^+, and Z^- are different *number systems*. However, only using the “largest” of these, *Z*, we are unable to solve all problems. For instance, if five persons are to equally share seven litres of water, then the volume of water each person will be given cannot be expressed in *Z*. Thus, we need an even “larger” system.

The set of all *rational numbers* consists of all numbers p/q where $p \in Z$ and $q \in Z$ but $q \neq 0$. This set is designated *Q*. Parenthetically, a set containing all objects of a certain “type” satisfying a certain condition may be described by specifying the “type” followed by the condition and a colon between them. Thus, we can specify *Q* writing

$$Q = \left\{ \frac{p}{q} : p \in Z, q \in Z, q \neq 0 \right\}.$$

All integers are rational numbers. (Let $q = 1$.) But also 0.5 (=1/2), 0.123456 (=1929/15625), 10.991 (=10991/1000), 0.333333333… (=1/3) and 0.28571428571428… (=2/7) exemplify rational numbers. Characteristic for all rational numbers is that their decimal expansion is either finite or periodic. Thus, one is always able to specify a rational number using only digits and (if necessary) a decimal point.

To emphasis that a decimal expansion is periodic, as in the expansion of 2/7 (where 285714 are infinitely repeated), one can overline the reoccurring numbers. For instance,

$$2/7 = 0.\overline{285714}.$$

Not all coordinates on a line, however, are rational. For instance, the number π, i.e. the ratio between the circumference and the diameter of a circle, cannot exactly be written p/q if $p \in Z$ and $q \in Z$. The set of all possible coordinates is called *the set of real numbers*, and is designated *R*. All rational numbers are real numbers, but also *irrational* numbers such as π, *e* and $\sqrt{2}$ (the diagonal of a square having the side 1). An irrational number can never be specified using only digits and a decimal point. Instead, *symbols* such as those above are used. It is always possible to compute a rational *approximation* to an irrational number, though. For instance we know that

$$\pi \approx 3.14159265358979324,$$

$$e \approx 2.71828182845904524 \text{ and}$$

$$\sqrt{2} \approx 1.41421356237309505.$$

Intervals

The set of all real numbers *x* such that $a < x < b$ can be written $]a,b[$. If, on the other hand $a \leq x < b$ we write $[a,b[$; if $a < x \leq b$ we write $]a,b]$ and if $a \leq x \leq b$ we write $[a,b]$.

These sets are called *intervals*. We realise that $R =]-\infty,+\infty[$. If, for instance, $a \leq x \leq b$ we can write that $x \in [a,b]$.

Prime numbers

A prime number is a number *n* such that $n \in N$, $n \geq 2$ and *n* cannot be divided by any other integer than ±1 and ±*n*, so that the result will also be an integer. If *P* is the set of all prime numbers, then

$$P = \{n : n \in N, n \geq 2, n/k \notin Z \text{ for all } k \in Z^+, k \notin \{1, n\}\}$$

Subsets

Let *A* and *B* be two sets. If all elements in *A* are elements in *B* as well, we say that *A* is a *subset* of *B*, which is written

$$A \subseteq B \text{ or } B \supseteq A;$$

the second expression is pronounced, “B is a *superset* of A”.

Please notice that the above expression permits that $A = B$. If we know that this is not the case, i.e. if there are elements in *B* that do not exist in *A*, we say that *A* is a *proper* subset of *B*.

$$A \subset B \text{ or } B \supset A.$$

Example 5

If we let

$$\text{sleepdays} = \{\text{Saturday}, \text{Sunday}\}$$

we realize that

$$\text{sleepdays} \subseteq \text{days}$$

and, more precisely,

$$\text{sleepdays} \subset \text{days}.$$

We realize that $Z^+ \subset N \subset Z \subset Q \subset R$. For a arbitrary set *A*, we also see that $A \subseteq A$ and $\varnothing \subseteq A$.

Union and intersection

Let *A* and *B* be two sets, and let the set *C* contain all elements being members of *A or B* (or both). Then we say that *C* is the *union* of *A* and *B*, or

$$C = A \cup B.$$

Then, let the set *D* contain all elements being members of *A and B*. Then we say that *D* is the *intersection* of *A* and *B*.

$$D = A \cap B$$

Example 6

Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$.

Then

$A \cup B = \{1,2,3,4,5,6\}$

and

$A \cap B = \{3,4\}$.

Example 7

Let $A = \{1,2,3\}$ and $B = \{4,5,6\}$.

Then

$A \cup B = \{1,2,3,4,5,6\}$

and

$A \cap B = \varnothing$.

Example 8

In a team of researchers in physics, there are 20 persons. Some of these are willing to work as teachers and some are experts in space technology. Let *A* be the set of those willing to teach and *B* the set of the experts in space technology.

At one occasion, a group of students are to be taught in space technology, and the research team is to provide with teachers, but only experts in the field. Then the set of suitable scientists equals $A \cap B$.

Difference

Let *A* and *B* be two sets. If the set *C* contains all elements being members of set *A* *except for* those elements that also exist in *B*, we say that *C* is the (set) difference between *A* and *B*, and write

$C = A - B$ or

$C = A \setminus B$.

We realize that $A - \varnothing = A$ and $A - A = \varnothing$ are valid for all sets *A*.

Example 9

Let A be the set of all students in a school class. At one occasion, some of the students in A are on a trip to a museum, and all other students are to study at home. Let B be the set of all students at the museum. Then $B \subseteq A$. Let C be the set of the students studying at home. Then $C = A - B$.

Example 10

Let A be the set of the entire staff at a university and let B be the set of all professors working there. Then all other personnel constitute the set $A - B$.

Universe and complement

If, at some occasion, we are studying sets of elements and *all* possibly relevant elements constitute the set U, then we say that U – at the moment – is the *universe*. For instance, studying a school class it is a natural approach to define U as the set of all pupils in that particular class. With the *complement* $\complement A$ to a set A, such that $A \subseteq U$, we mean the set $U - A$, i.e. all (at the moment) relevant elements that are *not* elements of A ("everything but A").

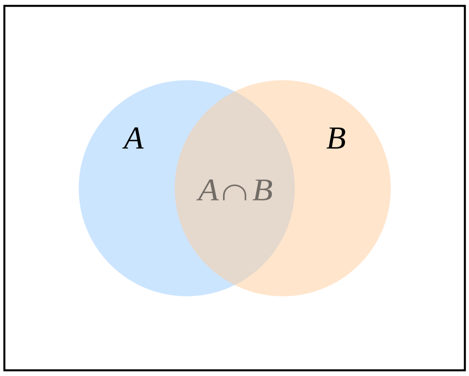
Example 11

Let us once again study the school class from example 9. If we define U as the set of all students in the class and A as the set of all students at the museum, then the students working at home constitute the set $\complement A$.

If U is a universe and A is a set satisfying $A \subseteq U$, we realise that $A \cup \complement A = U$ and $A \cap \complement A = \varnothing$.

Venn diagrams

To illustrate the relationships between a universe and its subsets, one can draw a Venn diagram. A Venn diagram is a rectangle corresponding to a universe and containing circles corresponding to subsets. A point in the diagram is an element being a member of all sets having a circle enclosing the point.



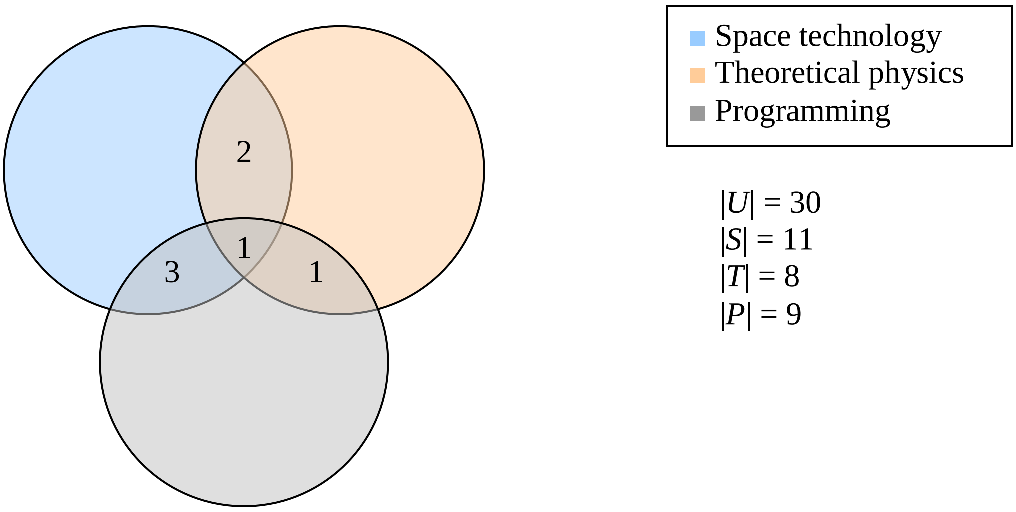
In the diagram above, two sets A and B are drawn. We have also drawn their intersection. Furthermore, all points inside either or both of the two sets are members of the union $A \cup B$. Thus, points outside the two circles constitute the complement $\complement(A \cup B)$.

Example 12

In a group of researchers consisting of 30 individuals, 11 are experts in space technology, eight are experts in theoretical physics, and nine are experts in programming. Two individuals are experts in both space technology and theoretical physics (but not programming), three are experts in both space technology and programming (but not theoretical physics), and one person is expert in both theoretical physics and programming (but not space technology). In addition, one scientist is expert in all three fields. Scientists not being experts in any field work as educators. How many of the scientists are experts in exactly one field, and how many of them are educators?

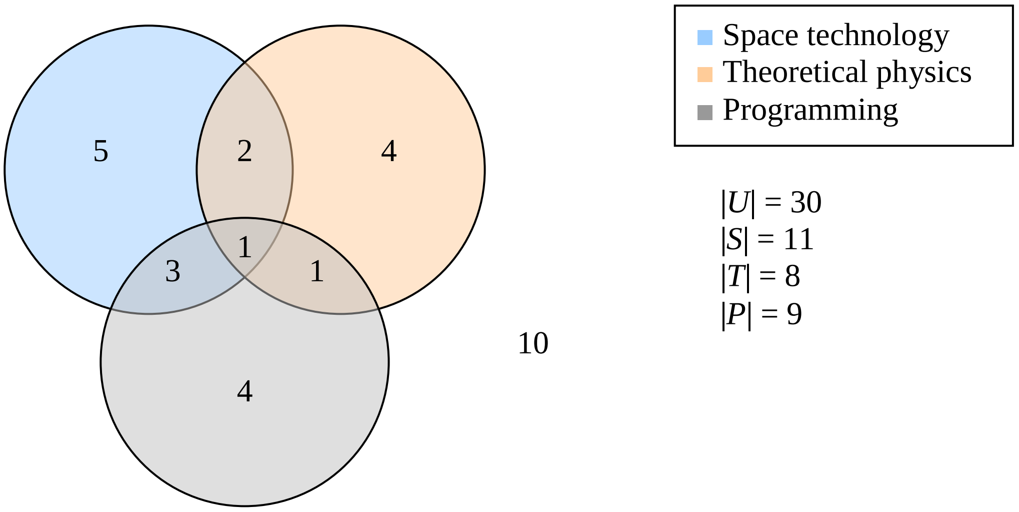
Solution:

This problem may be solved using a Venn diagram. We introduce the designation S for the set of all experts in space technology, T for the set of all experts in theoretical physics, and P for the set of all experts in programming. As universe U, we define the entire group of scientists. In the diagram below, we also mark the number of scientists belonging to each distinct area.



$|U| = 30$
$|S| = 11$
$|T| = 8$
$|P| = 9$

Overall, 11 individuals are experts in space technology. The only distinct subset of S not having any number assigned to it is the subset not constituting an intersection with another set. Thus, the number of scientists in this subset must equal $11-3-1-2=5$; five individuals are experts in *only* space technology. Analogously, we can fill in all missing numbers. We obtain the result below.



Answer: Five scientists are experts only in space technology, four are experts only in theoretical physics, four are experts only in programming, and ten are only working as educators.

Logic

We shall now study how we can *formalize* human intuitive perception of *logic*. Logic may be defined as the science about how to deduce (positive) *conclusions* from previous knowledge, *premises*. All conclusions must be absolutely true and unambiguous.

Proposition

A *proposition* is a piece of information telling us something about reality; a proposition is either *true* (correct) or *false* (incorrect). Hereafter, we shall use the number 1 instead of “true” and 0 instead of “false”.

Example 13

Let

p: “Earth is a planet”,
q: $5+5=15$ and
r: $2{,}5 \notin N$.

Then we know that

$p=1$,
$q=0$ and
$r=1$.

Operators

Operators are used to combine one or more propositions to a single, more “informative” proposition. A logical operator may be considered a function transferring one or two truth values to a new truth value.

Not

The *not* operator, designated ¬, converts a proposition to its opposite meaning. For instance, a false proposition will become true. The returned value is called a *negation*.

We can show the effect of the operator by using a *truth table*.

p	¬p
1	0
0	1

Example 14

Let p: “4 is a prime number”. According to the definition of prime numbers we realize that $p=0$, i.e. the proposition $\neg p=1$, or, in words, that “4 is *not* a prime number”.

We realize that $\neg(\neg p)$ has the same meaning as (*is equivalent to*) p, because the truth values for $\neg(\neg p)$ for all p equals the truth values of p.

p	$\neg p$	$\neg(\neg p)$
1	0	1
0	1	0

And

The *and* operator, designated $\wedge$, converts two operands (the one before the operator and the one after it) to a new proposition, called a *conjunction*. The conjunction is true if and only if both of the original truth values are true.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Example 15

The proposition “Linux is hungry and Linux is thirsty” is true if and only if Linux is both hungry and thirsty. Because if p: “Linux is hungry” and q: “Linux is thirsty”, then the entire proposition may be written $p \wedge q$, which is true if and only if $p = 1$ and $q = 1$.

Or

The *or* operator, designated $\vee$, also require two operands. The result of the operator is called a disjunction and is true if *either* of the operands is true, or if both of them are true.

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Example 16

The proposition “The student is a physicist or a mathematician” is true if the student is a physicist, a mathematician or both a physicist and mathematician at the same time. However, if the student is neither a physicist nor a mathematician, then the proposition is false.

The proposition “I am an amoeba or $1+1=2$” is always true, nevertheless, because the proposition $1+1=2$ always is true.

Exclusive or

Sometimes, using *or* in every-day communication, one mean that *exactly one* of the operands is true, but *not both*. This logical operator is called *exclusive or*, and is sometimes designated $\veebar$.

p	q	$p \veebar q$
1	1	0
1	0	1
0	1	1
0	0	0

Instinctively, we realize that the proposition $p \veebar q$ is equivalent to $(p \vee q) \wedge \neg(p \wedge q)$. In order more stringently to prove this, we can compare the truth values of the two expressions.

p	q	$p \vee q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
1	1	1	0	0
1	0	1	1	1
0	1	1	1	1
0	0	0	1	0

We see that the truth values for $p \veebar q$ and $(p \vee q) \wedge \neg(p \wedge q)$ always are identical to each other, which exactly is what we wanted to prove.

Using “or” in the every-day language, one sometimes mean logical *or* and sometime logical *exclusive or*. Thus, the mathematical logic is more unambiguous than the human (natural) language.

Example 17

If a computer technician says that there is something wrong with the keyboard *or* the software, it is likely that she means to use *logical or*, i.e. there is something wrong with *at least* one of the factors keyboard or software; there may be problems with both.

If a patient says to her physician that she want a drug administrated *orally* or *intravenously*, she likely means to use *exclusive or*; she does probably not wish to receive both an oral and an intravenous drug.

Implication

An implication is a proposition saying that another proposition is an inevitable consequence of yet another proposition. In pure English, for instance, we say that

”if p then q”

meaning that, if we know that the proposition p is true, we also know that q is true. Using symbols, we write

$p \Rightarrow q$ (“p implies q”).

This new proposition is true if both p and q are true. On the other hand, if p is true but q is false, then the expression is false as well, because p indeed has not implied q. However, if p is false, we cannot say anyting about the existence of the implication; instead we define the implication to be true. Thus, we obtain the following truth table.

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Example 18

Let

P(*x*): "*x* is a prime" and
E(*x*): "*x* is even".

Then we know that

$$\mathrm{P}(x) \wedge E(x) \Rightarrow x = 2 .$$

We can also use the alternative implication arrow, $\Leftarrow$, having exactly same meaning, but opposite "direction", i.e. the operands are shifted. Intuitively, we realize (and we can prove it by writing a truth table) that $p \Rightarrow q$ is equivalent to $\neg(p \wedge \neg q)$.

Moreover, we instantly realize that $[(p \Rightarrow q) \wedge p] \Rightarrow q$, $[(p \Rightarrow q) \wedge \neg q] \Rightarrow \neg p$ and $[(p \vee q) \wedge \neg p] \Rightarrow q$ always are true.

Example 19

Let

p: "It rains" and
q: "The ground becomes wet"

and assume that $p \Rightarrow q$ always is true.

Assume that $\neg q$. Then we know that $\neg p$, or, in other words: "If the ground does not become wet, then it does not rain".

Equivalence

If we knot that $p \Rightarrow q$ and that $q \Rightarrow p$ (i.e. $p \Leftarrow q$), then we can write this as an *equivalence*.

$$p \Leftrightarrow q$$

This means that p is true if q is true, and that q is true if p is true. Moreover, if p is false, then q must be false as well, and if q is false, then p must be false as well. We realize that p and q always have the same truth value, making them *equivalent* statements.

The equivalence above may also be read “p if and only if q”, or “p iff q”.

p	q	$p \Leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

As expected, we find that the truth values of the equivalence above are identical to the values of $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

Thus $(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$.

Example 20

We use the designations from example 18. Because also $x = 2 \Rightarrow \mathrm{P}(x) \wedge \mathrm{E}(x)$ the two statements are equivalent, and we can write

$$\mathrm{P}(x) \wedge \mathrm{E}(x) \Leftrightarrow x = 2.$$

Example 21

We have previously deduced that the statements $p \veebar q$ and $p \vee q \wedge \neg(p \wedge q)$ are equivalent. Thus $p \veebar q \Leftrightarrow p \vee q \wedge \neg(p \wedge q)$.

Equivalent equations

If the propositions A and B are equations and they are equivalent, then A and B share the same roots.

Solving an equation, an expression $f(x) = g(x)$ is usually, using different techniques, modified to look like $x = a$. It is important to make sure that each step produces an equivalent equation.

Example 22

Solve the equation $x^2 \sin x = 0$ with respect to *x*.

Solution:

We realize that

$$x^2 \sin x = 0 \Leftrightarrow \left(x^2 = 0\right) \vee \left(\sin x = 0\right) \Leftrightarrow x = n\pi \qquad \text{where } n \in Z .$$

Laws of logic

We can very easily realize that the following equivalences are true for all p and q:

$$\neg(\mathrm{p} \wedge \mathrm{q}) \Leftrightarrow (\neg \mathrm{p}) \vee (\neg \mathrm{q}) \tag{1}$$

$$\neg(\mathrm{p} \vee \mathrm{q}) \Leftrightarrow (\neg \mathrm{p}) \wedge (\neg \mathrm{q}) \tag{2}$$

These are *De Morgan's laws*. We can rephrase (1) as "if not both p and q are true, then at least one of them must be false" and (2) as "if it is not true that either of p and q is true, then both p and q must be false". If we more stringently wish to prove these equivalences, we need only to compare the truth values of each side of both equivalences.

Example 23

Let

p: "It rains" and
q: "The sun shines"

Then we can read the proposition $\neg(\mathrm{p} \wedge \mathrm{q})$ as

"It is not the case that it is raining and the sun shines".

According to De Morgan's law (1), this is equivalent to $(\neg \mathrm{p}) \vee (\neg \mathrm{q})$, or

"Either it does not rain or the sun does not shine (or neither)".

Unambiguousness

Notice again that the mathematical logic is more unambiguous than human natural language. The proposition

"It is not the case that it is raining and the sun shines".

may be interpreted (here: correctly) as

$\neg(\mathrm{p} \wedge \mathrm{q})$ (it is not the case that it is raining at the same time that the sun is shining)

but also (here: incorrectly)

$(\neg \mathrm{p}) \wedge \mathrm{q}$ (it does not rain but the sun shines).

It may be tempting to use clearifying brackets in human language as well. For instance we could write

“It is not the case that [it rains and the sun shines]”,

which is unambiguous[1].

---

[1] The alternative interpretation $(\neg p)\wedge q$ could be written

“[It is not the case that it rains] and the sun shines”.