Introductory Mechanics

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Preface

This document is intended to give the reader a simple introduction to classical mechanics, which describes and explains how objects move and interact with each other by means of forces. Classical mechanics describes how large amounts of the matter in the universe behave, and is also very intuitive and easily comprehensible. Classical mechanics, however, has some limitations; is it, for instance, unable to describe objects moving at speeds near the speed of light, c (approximately 300 000 km/s). At speeds significantly lower then c, however, it is an excellent approximation. Classical mechanics will undoubtedly always be correct, useful, intuitive and beautiful.

Physics

Physics is the most fundamental study of the universe. A physicist is a seeker of truth, who tries to find out how the universe and nature function at the very most elementary level. The acquired knowledge can, moreover, imply technological innovations which further develop the human being and her society. Almost all technology used on a daily basis is the result of physical knowledge. In order to discuss nature, we need a language to describe objects and phenomena. We need to define measurable properties, called *quantities*, and which are given in relation to defined values, *units*. In the following table, we define the quantities used in this document.

Quantity	**Description**	**Unit**
Distance, displacement, position (s)	Distance, displacement or position in space	metres (m)
Time (t)	Length, position or distance in time	seconds (s)
Mass (m)	Property of matter; amount of matter	kilograms (kg)
Charge (Q)	Property of matter	coulombs (C)

Prefixes

In order to deal with very large and small numbers (for instance the huge distance between the sun and the earth or the minuscule distance between the nucleus of an atom and its surrounding electron shells), we use prefixes to the units. For instance, we write 1 kilometre (km) instead of 1 000 metres (m). The table below defines the prefixes used in this document.

Symbol	**Name**	**Factor**
T	tera	10^{12}
G	giga	10^{9}
M	mega	10^{6}
k	kilo	10^{3}
d	deci	10^{-1}
c	centi	10^{-2}
m	milli	10^{-3}
µ	micro	10^{-6}
n	nano	10^{-9}
p	pico	10^{-12}

Some mathematical symbols

Below we describe some of the mathematical symbols used in this document.

Symbol	**Meaning**	**Usage**
$\mapsto$	from … to	$x \mapsto y$ means that *y* is a function of *x*
$\in$ [/] … [/]	is an element of interval notation	$a \in A$ means that *a* is an element of the set *A* $a \in [A, B]$ means that $A \le a \le B$, whereas $a \in [A, B[$ means that $A \le a < B$
$\perp$	is perpendicular to	$a \perp b$ means that *a* is perpendicular to *b*, i.e. that the angle between *a* and *b* equals 90°
‖	is parallel to	*a* ‖ *b* means that *a* is parallel to *b*, i.e. that the angle between *a* and *b* equals 0° or 180°

What is motion?

It is difficult, in any simple way, to define the fundamental quantities described above; instead, we rely on our intuition of them. In order to define motion, we can think as follows: Let an object be located at a point *A* in space at a time t_0. If the object later on at a time t is located at another point *B* in space, we say that the object has *moved* the distance between *A* and *B* during the time $t - t_0$, or that it has performed a *motion*. Using the units “metre” and “second” to measure lengths and time intervals, with the help of mathematics, we are able to derive several useful relationships between these quantities.

Linear motion

We shall begin with studying simple examples of how objects move; we are to determine the relationships between their displacements, velocities and accelerations.

Distance, *displacement* or *position* (s) describes a distance, change of position or just position in space, often how long distance an object has travelled, and is given in relation to the metre (m) unit. By position, we often mean the distance to the origin (0) on a line representing different locations in space. *Time* (t) describes a distance in time, often the duration of a phenomenon, and is measured in seconds (s). We can also mean the (time) distance to the origin (0) on a timeline.

Velocity

The *velocity* (v) describes how long distance an object moves during one unit of time, i.e. how fast it moves, and is measured in the metres per second (m/s) unit. Often the velocity of an object is a none-constant function of time. The average velocity $\bar{v}$, between two fixed positions in time, however, specifies how fast the object in average has moved between the two points in time. If a car, for instance, is at the point s_0 at the time t_0 and at s at time t, the average velocity is equal to

$$\bar{v} = \frac{s - s_0}{t - t_0}.$$

If we instead write Δs for the displacement $s - s_0$ and Δt for the change in time $t - t_0$, we obtain the simpler expression

$$\bar{v} = \frac{\Delta s}{\Delta t}.$$

If it is obvious that we mean a *change* in space and time, the delta signs (Δ) can be omitted. The mean bar above v can be omitted as well.

The velocity at *a particular moment* is called *the instantaneous velocity*. The instantaneous velocity can not be measured as we need two observations at different times in order to measure the associated change of position; during exactly zero seconds, objects do not move at all. It is however possible to approximate the instantaneous velocity at any given time t with the average velocity during a very short period of time containing the point of time t. This is how speedometers in cars work. Mathematically, the instantaneous velocity is defined as the limit of the average velocity as $\Delta t \to 0$, i.e.

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

If we know a function $t \mapsto s$, the instantaneous velocity equals the derivative of the displacement with respect to time:

$$v = \frac{ds}{dt}$$

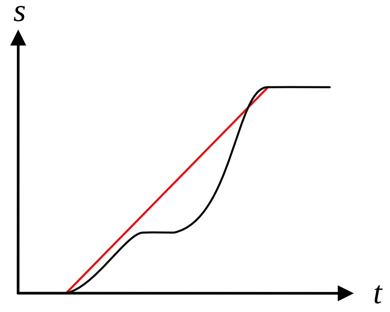
If we know the mean velocity and the time of an object's motion, we can very easily compute the travelled length (displacement) by using of the equation $s = \bar{v}t$.

If an object travels with a constant velocity, the motion is said to be uniform. If the velocity, on the contrary, is variable with respect to time, the motion is said to be accelerated. If the velocity is positive, the object moves in a forward direction; if the velocity is negative, the object moves backwards, *relative to the direction we have defined to be positive*.

If we plot the graph (of a journey with a car, for instance) with the displacement on the *y* axis and time on the *x* axis, then the slope of the tangent at any moment equals the instantaneous velocity at that particular moment.

Example 1

Amanda takes her car to her office. On her way there, she has to stop at a crossing. The graph below illustrates her journey. We see that her average velocity after the stop was somewhat greater than the average velocity before it. The slope of the red line equals the average velocity of the entire journey. Apparently, the average velocity equals the *constant* velocity that, during the same amount of time as the real journey, results in the same displacement in space as well.



Example 2

We dricing a car between two towns located 370 km from eachother. We drive with a constant velocity of 90 km/h. How long will the journey take?

Solution:

$$s = 370\ \text{km}$$

$$v = 90\ \text{km/h} = 25\ \text{m/s}$$

$$s = vt \Leftrightarrow t = \frac{s}{v}$$

$$t = \frac{370000\ \text{m}}{25\ \text{m/s}} = 14800\ \text{s} \approx 4\ \text{h}\ 7\ \text{min}$$

Answer: The journey will take slightly more than four hours.

Example 3

Two stones approached each other freely in space. At one moment the distance between the stones was 400 m. They collided 80 seconds after that moment. At what velocity did they collide?

Solution:

$s = 400\,\text{m}$

$t = 80\,\text{s}$

$v = \frac{s}{t} = \frac{400\,\text{m}}{80\,\text{s}} = 5\,\text{m/s}$

Answer: The stones collided at a velocity of 5 m/s.

Acceleration

The *acceleration* (a) states how fast a velocity changes with respect to time and is measured in metres per second squared (m/s^2). (Please note that $1\,\text{m/s}^2 = 1(\text{m/s})/\text{s}$.)

The average acceleration is defined as change of velocity per unit of time:

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

The instantaneous acceleration is defined analogously to instantaneous velocity:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

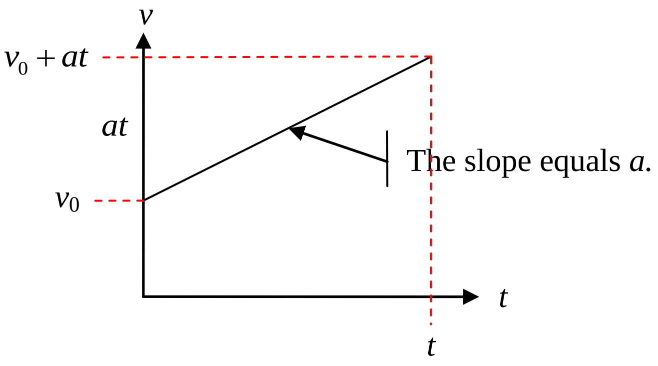
If we know a function $t \mapsto v$, the instantaneous acceleration can be defined as the time derivative of the velocity:

$$a = \frac{dv}{dt}$$

An accelerated motion with constant acceleration is called a uniformly accelerated motion. An accelerated motion with variable acceleration is called a jerked acceleration. If the acceleration is positive, the velocity of the object increases; if the acceleration is negative, the velocity decreases.

If we know the average acceleration (or the constant acceleration) during a period of time, we can determine the total change in velocity by using the equation $\Delta v = at$. If an object moving with initial velocity v_0 is accelerated with the constant acceleration a during the time t, the final velocity will be $v = v_0 + \Delta v = v_0 + at$.

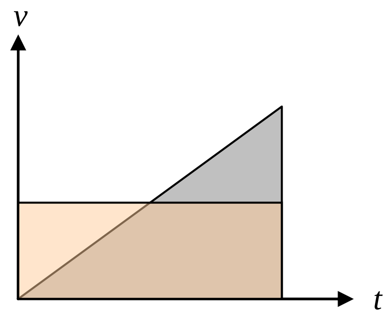
If we make graph with the velocity on the y axis and the time on the x axis, the slope of the tangent at any moment will be equal to the instantaneous acceleration (at that moment) and the area (integral) below the line will be equal to the total displacement in space. On an a versus t plot, the integral will be equal to the total change in velocity. The graph below shows a uniformly accelerated motion with initial velocity v_0. The acceleration is a.



Is it possible to calculate the total displacement in space if the object is accelerated uniformly (and we do not have access to a graph)? We now want to find the displacement as a function of the initial velocity v_0, the constant acceleration a and the duration t of the motion.

First, we recall that $s = \bar{v}t$.

The average velocity $\bar{v}$ of a uniformly accelerated motion equals the arithmetical mean of the first velocity v_0 and the last velocity v, i.e. $\bar{v} = \frac{1}{2}(v_0 + v)$. In the diagram below, in witch the height of the rectangle equals the arithmetical mean of the first and last v value of the triangle, we can see that this is indeed true, since the area of the rectangle equals the area of the triangle (and hence they represent equal displacements during equal time).



Thus we may write

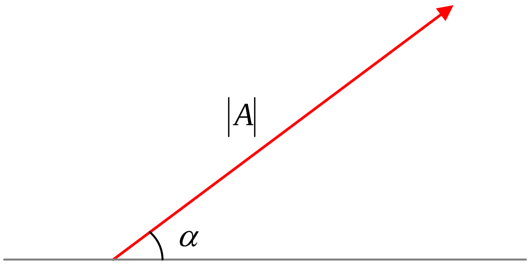
$$s = \tfrac{1}{2}(v_0 + v)t$$
$$s = \tfrac{1}{2}(v_0 + v_0 + at)t = \tfrac{1}{2}(2v_0 + at)t = v_0 t + \tfrac{1}{2}at^2.$$

If the object initially was at rest, i.e. if $v_0 = 0$, the equation turns out to be particularly simple:

$$s = \tfrac{1}{2}at^2$$

Vector quantities

A *vector quantity* is a quantity represented by an arrow defined by both its length and its direction (but not its location). A quantity without a direction, i.e. an ordinary number, is called a *scalar quantity*. The length of a vector A (to emphasis that a quantity really is a *vector* quantity, we can use the notation $\vec{A}$) is called its *magnitude* and is written $|A|$ (or just *A* if the direction is irrelevant in the discussion). The direction can be stated in different ways (with means of geographical directions, angle to a specified line etc.). Below we exemplify with the vector *A*.



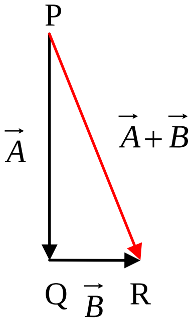
A displacement may be reprecented using a vector: we can for instance walk eight kilometres (the magnitude) to the west (the direction). Velocity is a vector too, since a moving object both travels some metres per second (the magnitude) and does this in a certain direction. Even the acceleration a vector, as it both changes the velocity with a certain amount of metres per second squared, and does this in a certain direction. So far, however, we have only studied accelerations that directly have increased or decreased the velocity in the velocity's very own direction. Later on, we will study cases where the acceleration is not parallel to the velocity (for instance if an object has a positive velocity to the right and is accelerated downward, like a bullet influenced by gravity).

The negative of a vector

The *negative of vector* $-\vec{A}$ to $\vec{A}$ has the same magnitude as $\vec{A}$ but has the opposite direction (is rotated 180°). If you, for instance, walk minus five steps forward, then you walk five steps backward.

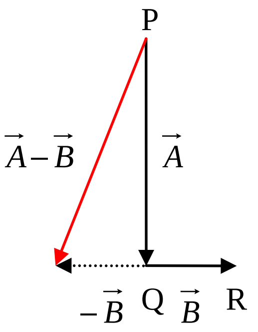
Addition and subtraction

Two vectors can be *added* to each other. Imagine for example that Amanda, standing at the point *P*, walks five kilometres straight to the south to a point *Q* and then two kilometres to the east to a point *R*. We represent the both displacements with the vectors *A* and *B*. It is practical to define the vector sum $\vec{A}+\vec{B}$ as the vector describing the total displacement from *P* to *R*.



We realize that we get the sum of two vectors by, having the second vector starting where the first ends, drawing a new arrow (vector) from the initial (starting) point of the first vector to the terminal (ending) point of the second vector. From the definition of vector addition, we realize that the operation is commutative, i.e. that $\vec{A}+\vec{B}=\vec{B}+\vec{A}$, which also applies to scalars.

Subtraction of vectors is performed by adding the first vector to the negative of the second vector, $\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$, also just like what applies to scalars.

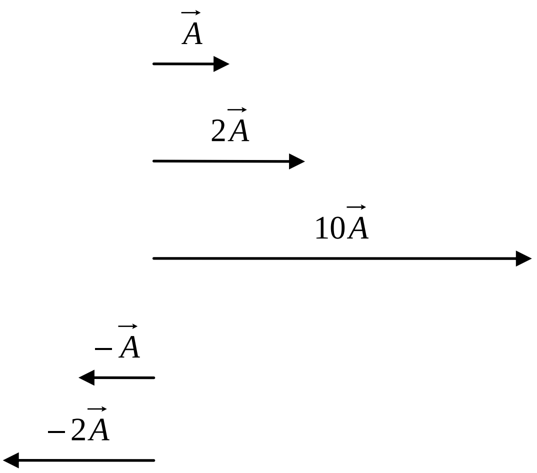


If $\vec{C}=\vec{A}+\vec{B}$ then $\vec{A}$ and $\vec{B}$ are called *components* to $\vec{C}$ which is called the *resultant* to $\vec{A}$ and $\vec{B}$. Please note that our definition of vector subtraction implies that $\vec{A}-\vec{B}=\vec{C}\Leftrightarrow\vec{A}=\vec{B}+\vec{C}$, which also applies to scalars.

Multiplication with scalar

With the product $k\vec{A}$ we mean the vector that has the same direction as $\mathrm{sgn}(k)\cdot\vec{A}$[1] and whose magnitude equals the product of the magnitude of k and the magnitude of $\vec{A}$, i.e. $\left|k\vec{A}\right|=\left|k\right|\left|\vec{A}\right|$.

Note that the direction of the product is the opposite of the one of $\vec{A}$ if k is negative.



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[1] The sgn (sign or *signum*) function returns the sign of the argument, i.e. $\mathrm{sgn}(x)=\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x=0\\ -1 & \text{if } x<0\end{cases}$

Also notice that, if k is a natural number, then $k\vec{A} = \underbrace{\vec{A} + ... + \vec{A}}_{k \text{ terms}}$, which also applies to scalars.

Velocity as a vector

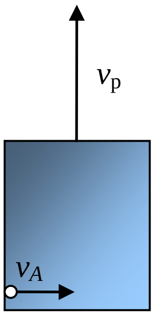
Velocity is a vector. A car can for instance travel 90 km/h to the north. The magnitude of a velocity (here 90 km/h) is called its *speed*. A speed does therefore not include any information of the direction of the motion. Being vectors, velocities can be added to each other.

Example 4

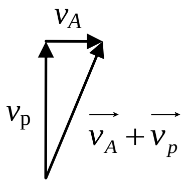
A large floating platform transports humans over a watercourse. The platform travels straight to the north at a velocity of 6.0 m/s. Being near the west edge of the platform, Amanda starts walking straight to the east with a velocity of 2.5 m/s (relative to the platform). At what velocity does Amanda move relative to the surrounding area (the city)?

Solution:

We draw the platform, mark Amanda's position and draw their velocity vectors.



A vector is defined by its length and direction – not by its position – and therefore we are allowed to move them so that they lie after one another.



In one second the platform moves $|v_p|$ metres to the north whereas Amanda moves $|v_A|$ metres to the east, relative to the platform. In one second, therefore, Amanda moves $|\vec{v_A} + \vec{v_p}|$ metres relative to the surroundings in the same direction as the vector $\vec{v_A} + \vec{v_p}$. Hence, her total velocity equals the vector sum $\vec{v_A} + \vec{v_p}$. Let us call the sum v.

We now want to determine both the magnitude and direction of v. The magnitude is given by Pythagoras theorem:

$$|v| = \sqrt{v_A^2 + v_p^2} \approx 6.5 \text{ m/s}$$

The direction is given by arctangent:

$\alpha = \arctan \frac{v_A}{v_p} \approx 23°$

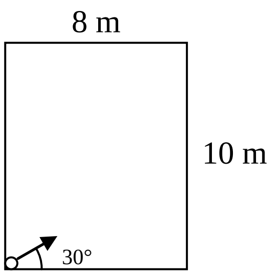
Answer: Amanda moves with the speed 6.5 m/s in the direction 23° clockwise from the platform's direction of motion.

Concerning velocities, *separation into components* is often very useful. Separation into components means that we separate a vector $\vec{C}$ into two components $\vec{A}$ and $\vec{B}$ such that $\vec{C} = \vec{A} + \vec{B}$.

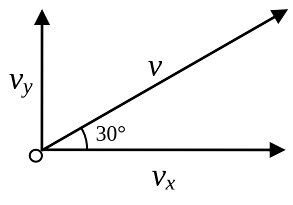
Example 5

A room has the dimensions 8 times 10 metres. Standing in the southwest corner of the room, Linux walks with a constant velocity of 2 m/s, making an angle of 30° with the southern wall of the room. After how long does he reach the eastern wall of the room?

Solution:



Let us magnify the corner in which Linux stands and separate Linux' velocity vector v into a horizontal component v_x and a vertical component v_y such that $\vec{v_x} + \vec{v_y} = \vec{v}$.



Notice that this choice of separation results in two right triangles. With this separation we are able to compute Linux' horizontal velocity (how fast he approaches the eastern wall) as well as his vertical velocity (how fast he approaches the northern wall). We compute the magnitude of the horizontal velocity v_x:

$$|v_x| = v \cos 30° \approx 1.7 \text{ m/s}$$

The width of the room $s = 8\,\text{m}$, which implies that $t = \frac{s}{v_x} = 4.6...\,\text{s} \approx 5\,\text{s}$.

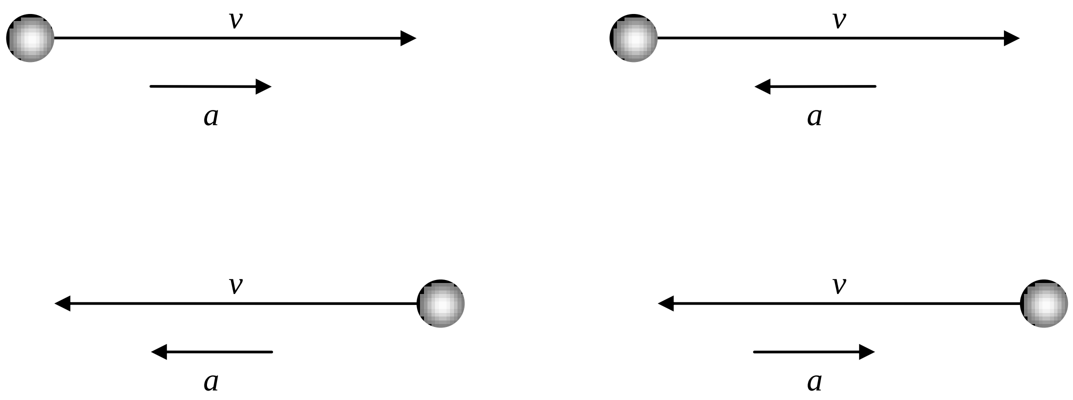
Answer: Linux reaches the eastern wall of the room after somewhat less than 5 seconds.

Acceleration as a vector

The acceleration vector tells us how fast the velocity changes with respect to time. If $a = 0$ then the object moves with constant velocity. If $a > 0$ the velocity increases and if $a < 0$ the velocity decreases. Please notice, that if the velocity is negative, i.e. if the objects moves backwards relative to the chosen positive direction, then $a > 0$ means that the speed (the magnitude of the velocity) decreases, whereas $a < 0$ means that the speed increases. In mechanical discussions, it is therefore very important that one carefully pays attention to the direction of the vectors.

It is possible that acceleration makes the velocity of an object change its direction. If, for instance, an object that moves with a positive velocity to the right is accelerated to the left (i.e. the acceleration to the right is negative) the speed will decrease. Eventually the speed will reach 0 when the object momentarily is at rest. If the acceleration remains, the object will thereafter receive a negative velocity, i.e. it will continue to move to the left, after which the speed to the left will increase.

The speed increases	**The speed decreases**



In the cases where the speed decreases it will eventually reach 0, after which the velocity will change its direction and the speed will begin to increase.

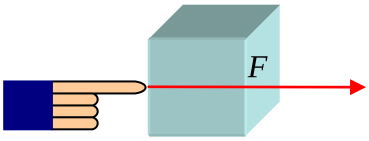
If the acceleration is not parallel to the velocity, the object will follow a *curved* path. We will later on study this more in detail. Please note, that we yet have not discussed *why* objects accelerate at all; an object can not accelerate by itself, but a *force* is needed to accelerate an object. Forces will be discussed in the next chapter.

Forces

A fundamental property of matter is that particles and larger objects are unable to change their velocity by themselves. If an independent object in space hovers at rest, it will remain to hover at rest. If it on the contrary is moving, it will continue to move with the same, constant, speed and with the same direction. In order to change the velocity of an object, we must "force" it to. An object changes its velocity, i.e. begins to accelerate, when it is affected by a *force*. We can also say that a force *acts* on the object. One measures forces in the Newton (N) unit after the "founder" of classical mechanics, Sir Isaac Newton.

Force is also a vector. When one draws a force vector arrow on an object, the arrow should start where the force in the real world acts on the object. This is, of course, only possible in those cases where the force really acts on a particular point. We are now, very briefly, about to study forces *intuitively*, in order to increase the reader's understanding of them, after which we will give the phenomena more careful mathematical descriptions.

We imagine an object *A* which hovers freely in a spaceship. Relative to the room, the object hovers at rest, i.e. $v = 0$. By *pushing* the object to the right (by pressing on its left side), i.e. applying a rightward force *F* on the object, the object will accelerate to the right. When we stop pushing the object, the acceleration will disappear and the object will preserve the received positive velocity to the right. The velocity will remain constant until the object is affected by another force, for instance when colliding with the right wall of the room.



Resulting force

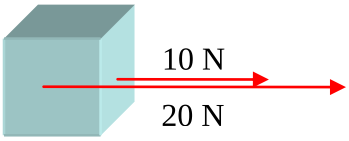
If we, on the object *A*, also push somewhat weaker on the *right* side of the object, the acceleration that *A* receives will be somewhat lower. If we instead push equally hard on its right side, no acceleration at all will be given to *A*; it will continue to hover at rest. If we push harder on the object's right side than on its left side, it will accelerate to the left. This suggests that also forces can be added and leads us to the following conclusion:

If several forces, *component forces*, act on one and the same object, the vector sum of all forces is called the *resulting force* and is often designated F_{res}. The imagined resulting force has the same effect on the object as all of the added component forces together.

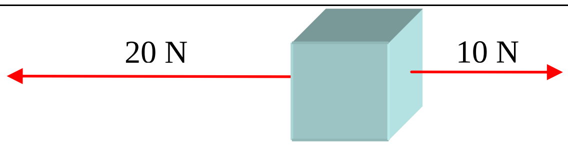
Example 6

Compute and draw the resulting force that acts on the object. In which direction will the object accelerate?

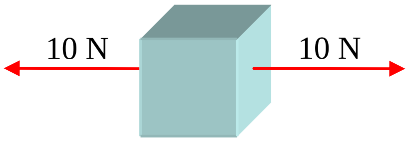
a)



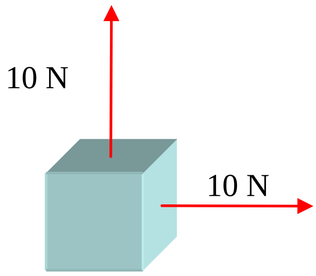
b)



c)

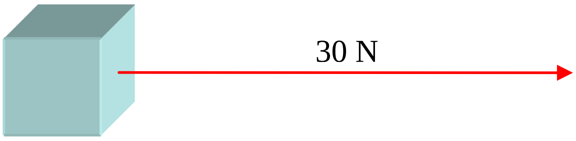


d)



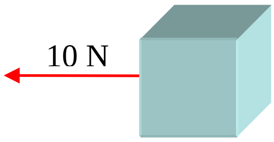
Solution:

a)



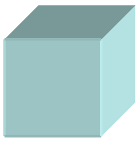
The object will accelerate to the right.

b)



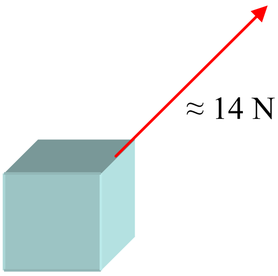
The object will accelerate to the left.

c)



The object receives no acceleration at all.

d)



The object will accelerate in an up-right direction; the angle to the horizontal component is 45°.

Newton's first law

As we have seen earlier, an object preserves its velocity until a resulting force acts on it. If no resulting force acts on an object, it will remain at rest if it from the beginning was at rest, or preserve its constant velocity if it from the beginning was moving.

Hence, an object obtains acceleration if and only if the resulting force acting on the object is not equal to zero.

$F_{res} = 0 \Leftrightarrow a = 0$ (Newton's first law of motion)

If we, being in space, push on an object at rest, the object will obtain an acceleration increasing its speed. The velocity will get the same direction as the force. When we stop pushing, the object will preserve its velocity obtained from the acceleration. If we, being on the earth's surface, sit in a car and give the car a forward acting force by using its engine, the car will – as with the object above – receive a speed-increasing acceleration. When we let go of the throttle, however, the car will slow down. The force that is slowing down the vehicle is mainly the friction against the road (the tires are hitting the rough surface of the road) although air resistance also contributes.

Newton's second law

A resulting force makes an object accelerate. It is reasonable to believe, that a greater force results in a greater acceleration, and that greater forces are required in order to accelerate larger objects. This is true, and it is the meaning of Newton's second law:

$F_{res} = ma$ (Newton's second law of motion)

or equivalent

$$a = \frac{F_{res}}{m}$$

where F_{res} is the resulting force acting on the object with mass m which therefore receives the acceleration a. According to the formula above $1\,\mathrm{N} = 1\,\mathrm{kg m/s^2}$ and the force 1 Newton is defined as the force that gives an object with mass $m = 1\,\mathrm{kg}$ an acceleration $a = 1\,\mathrm{m/s^2}$. Please note that the definition of force depends on the definitions of mass, length and time, and that the definition of the Newton unit as a consequence depends of the definitions of the kilograms, metres and seconds units.

We realize that Newton's first law is an implication of Newton's second law as $m \neq 0$. We also notice that the acceleration has the same direction as the resulting force because m is a scalar quantity.

Example 7

In space, a 5 kg block is hovering freely. When the block is at rest relative to the spaceship, we push on its right side with a constant force of 2 N. We continue to push with this force in 0.5 seconds. What velocity has the block attained when we let go of it?

Solution:

$a = \frac{F_{res}}{m} = 0.4\ \text{m/s}^2$

$t = 0.5\ \text{s}$

$v_0 = 0 \Rightarrow v = at = 0.2\ \text{m/s}$

Answer: The block will have attained a velocity of 0.2 m/s.

Example 8

A block with a mass of 0.25 kg lies at rest on a table. We push on the block with the constant force $F = 0{,}8\ \text{N}$. This makes the block move on the table with a constant velocity. Determine the friction force F_μ acting on the block, ignoring air resistance.

Solution:

$a = 0 \Leftrightarrow F_{res} = 0$

$F_{res} = F + F_\mu \Leftrightarrow F_\mu = -F = -0.8\ \text{N}$

Answer: The friction force $F_\mu = -0.8\ \text{N}$.

Newton's second law also applies to components of force and acceleration. If $F = ma$, $\vec{F} = \vec{F_x} + \vec{F_y}$ and $\vec{a} = \vec{a_x} + \vec{a_y}$, then we have

$$F_x = ma_x$$

and

$$F_y = ma_y.$$

Newton's third law

If an object affects another object with a force, then the other object also affects the first object with an equally great, oppositely directed force of the same sort. This principle is called Newton's third law.

If an object A acts on an object B with a force F, then B acts on A with force $-F$. (Newton's third law of motion)

If, for instance, a tennis player hits a tennis ball with her racket, then the racket acts on the ball with the force F, accelerating the ball in a forward direction. But an equally great but opposite force $-F$ acts on the racket from the ball. Since the player is holding the racket (and hence is applying an appropriate force on it during the racket-ball collision), however, it will not receive any remarkable acceleration backwards. (Notice, however, that the forward positive velocity of the racket of course must decline at some point during or directly after the collision, to which the collision with the ball might contribute.)

Gravity

All objects with mass affect each other with attracting forces: gravitational forces. The sun keeps the earth and the other planets in our planetary system in elliptical orbits around it by means of gravitational forces, and the earth keeps the moon in its orbit around the earth with gravitational forces. On the surface of the earth all objects, even the atmosphere, is kept down by the earth's gravitational field. If we, standing on the earth's surface, hold an apple and then let go of it, it will fall to the ground because of the gravitational force acting on it.

Newton discovered that the gravitational force F_G between two bodies is proportional to the product of the objects' masses and inversely proportional to the square of the distance between them,

$$F_G = G\frac{m_1 m_2}{r^2} \qquad \text{(Newton's law of universal gravitation),}$$

where m_1 and m_2 are the mass of each body (respectively) and r is the distance between the objects' centres of mass. The constant of proportionality $G \approx 6.67\times10^{-11}\ \mathrm{Nm^2/kg^2}$ is called the gravitational constant.

Example 9

The mass of the earth is 5.97×10^{24} kg and the mass of the sun is 1.99×10^{30} kg. The distance between the sun and the earth is at one occasion 147 Gm. Determine the gravitational force on the earth from the sun and the gravitational force on the sun from the earth.

Solution:

$$F_G = G\frac{5.97\times10^{24}\ \mathrm{kg}\cdot 1.99\times10^{30}\ \mathrm{kg}}{\left(147\times10^{9}\right)^2\ \mathrm{m^2}} \approx 3.67\times10^{22}\ \mathrm{N}$$

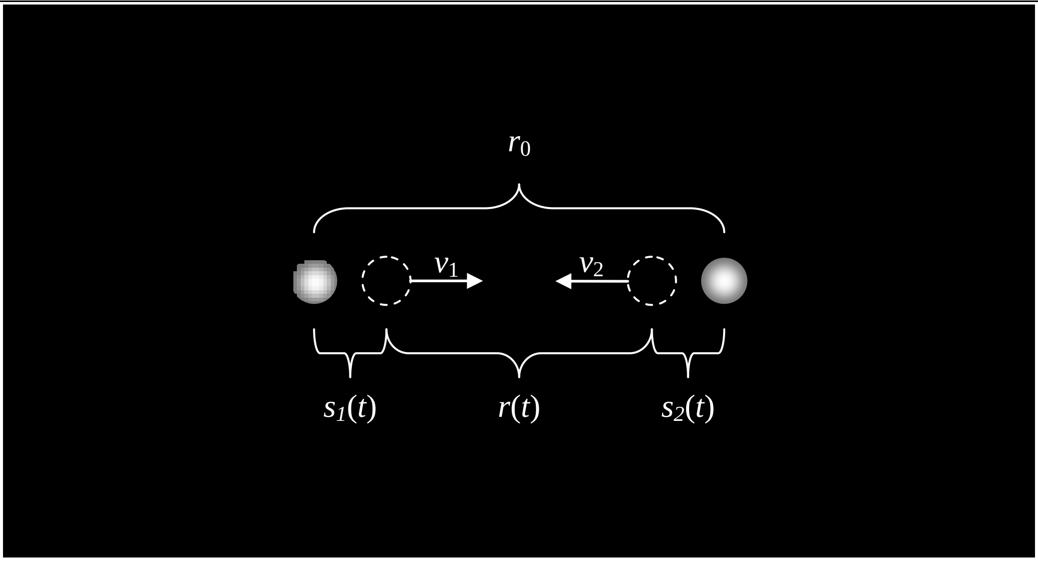
Answer: The sun affects the earth with the force 3.67×10^{22} N directly towards the sun. The earth affects the sun with the force 3.67×10^{22} N directly towards the earth.

Example 10

Two stones with masses m_1 and m_2 are hovering freely in space, not close to any other massive celestial body. They are at rest relative to each other and the distance between them equals r_0. After how long do they collide with each other?

Solution:

Let $r(t)$ denote the distance between the stones at time t. The initial distance $r(0) = r_0$. The force on each stone at time t is therefore $F(t) = G\frac{m_1 m_2}{r(t)^2}$ and the accelerations of the stones will be $a_1(t) = G\frac{m_2}{r(t)^2}$ and $a_2(t) = G\frac{m_1}{r(t)^2}$, respectively. At time t the stones have travelled the distances $s_1(t)$ and $s_2(t)$, respectively. See the illustration below.



We realize that the stones' velocity towards each other increases with time, as does their acceleration.

According to the illustration we have $r(t) = r_0 - s_1(t) - s_2(t)$. After differentiating the equation we get

$$r'(t) = -s_1'(t) - s_2'(t)$$

and again

$$r''(t) = -s_1''(t) - s_2''(t) = -a_1(t) - a_2(t) .$$

Insertion of the expressions of a_1 and a_2 results in

$$r''(t) = -G\frac{m_2}{r(t)^2} - G\frac{m_1}{r(t)^2} = -G(m_1 + m_2)r(t)^{-2} .$$

If we let $k = -G(m_1 + m_2)$, we get

$$\begin{cases} r'' = kr^{-2} \\ r(0) = r_0 \\ r'(0) = 0 \end{cases} .$$

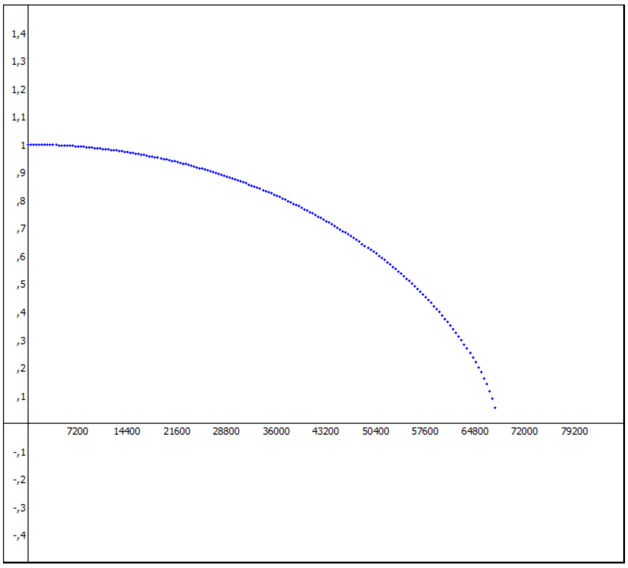
This initial-value problem can easily be solved numerically.

Example 11

Let the stones from example 10 from the beginning be on the distance $r_0 = 1\,\text{m}$ from each other, and presume that they have equal masses $m_1 = m_2 = 2\,\text{kg}$. After how long do they collide with each other?

Solution:

Let $k = -G(m_1 + m_2) = -4G$. Numerically we are able to plot the function $t \mapsto r$. By using the author's software AlgoSim (www.rejbrand.se/algosim), we obtain the following graph.



We see that the stones collide after approximately 18.8 hours.

Gravity near the surface of a celestial body

All objects near the earth or any other celestial body is attracted to this by gravitational forces. The gravitational force acting on an object is called the object's *gravity*.

The gravitational force on an object is, according to Newton's law of universal gravitation, proportional to the mass of the object. Thus, for every given distance to a particular celestial body, we can compute the fixed, definite force per kilogram of mass. This quantity is called the gravitational field strength, designated g and must have the unit N/kg.

Now we want to compute the gravitational field strength at the earth surface, where we live.

$$g = \frac{F_G}{m} = \frac{G\frac{m_e m}{r^2}}{m} = G\frac{m_e}{r^2}$$

where m_e is the earth's mass and r is the earth's radius. As $m_e = 5.9736 \times 10^{24}$ kg and $r = 6373$ km we get $g \approx 9.81$ N/kg. The earth is not a perfect sphere, but a spheroid, which implies that g not exactly – buy yet approximately – is equal everywhere on the surface of the earth. The gravity of an object with mass m is therefore $F_G = mg$ at the surface of the earth.

Example 12

A bag has the mass $m = 12$ kg. With what force F must we hold it when it is at rest in the air?

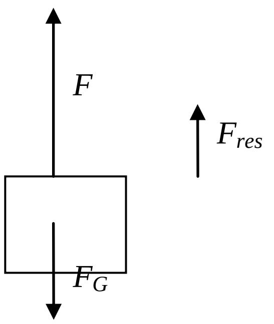
Solution:

According to Newton's first law $|F| = |F_G| = mg = 12\,\text{kg} \cdot 9.81\,\text{N/kg} \approx 118\,\text{N}$. Note that the gravity of the bag, and therefore also the force with which we must hold it becomes smaller if we stand on the surface of a less massive planet, for instance Mars.

Example 13

The bag from example 12 stands on the floor. We lift it upwards with a force giving the bag the acceleration $a = 1\,\text{m/s}^2$ upwards (we choose positive direction upwards). With what force F must we lift it? (We ignore air resistance.)

Solution:



$$\begin{cases} F_{res} = ma \\ F_{res} = F + F_G \end{cases}$$

$$ma = F + F_G \Leftrightarrow F = ma - F_G$$

$$F \approx 130\,\text{N}$$

(Please notice that $F_G < 0$.)

Answer: We must lift the bag with a force of 130 N.

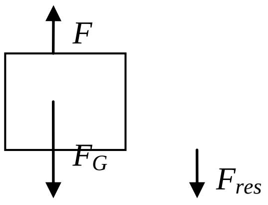
Notice:

If we after a while are lifting the bag with a constant, upward-pointing velocity, then the lifting force will again be equal to the gravity of the bag, i.e. approximately 118 N.

Example 14

The bag from example 12 is hold at rest in the air. We then lower it by holding it in such a way that it receives an acceleration of $1\,\text{m/s}^2$ downwards. (We still regard the upward direction as positive, and hence $a = -1\,\text{m/s}^2$ in that direction.) With what force F must we then hold the bag?

Solution:



$$\begin{cases} F_{res} = ma \\ F_{res} = F + F_G \end{cases}$$

$$ma = F_G + F \Leftrightarrow F = ma - F_G$$

$$F \approx 106\ \text{N}$$

Answer: We must hold the bag with a force of 106 N.

Acceleration due to gravity

If we, holding an object above the ground, let go of it in a gravitational field (a place where gravitational forces act on masses) so that no other forces than the gravitational force acts on the object, it will accelerate and is said to be in a *free fall*. If we let go of a stone some metres above the surface of the earth, it will not fall freely, because also the air resistance will be acting on it (upwards). The stone will obtain a somewhat lower acceleration than if there would not have been any air resistance. If the stone is in vacuum, however, it will of course fall freely towards the ground.

We now want to determine the acceleration *a* that an object with mass *m* receives if it is allowed to fall freely near the surface of the earth.

$$a = \frac{F_{res}}{m} = \frac{F_G}{m} = \frac{mg}{m} = g$$

We see that the acceleration that the object obtains is equal to the gravitational field strength at the location. This applies generally (as the formulas suggest) – and hence not only here on earth. We also see that all objects, regardless of their mass and gravity, fall with the same acceleration. If we ignore air resistance, a lead weight with a mass of 10 Mg will fall with the same acceleration – 9.82 m/s^2 – toward the earth as a piece of cotton-wool with a mass of 0.2 grams. This law is often called Galileo’s law.

Example 15

Amanda throws a ball right up in the air. After the time $t_{tot} = 1.4\ \text{s}$, she catches it again. We ignore air resistance.

a) Which acceleration did the ball have during the throw?

b) How long time does the journey up to the maximum height take?

c) What is the maximum height of the ball?

d) What velocity did the ball have when it left her hand?

e) What velocity did the ball have at its maximum height?

f) What velocity did the ball have when she caught it again?

g) Plot the functions $t \mapsto s$, $t \mapsto v$ and $t \mapsto a$ for $t \in [0,1.4]$, where *s* is the height over her hand.

Solution:

a) The ball is only affected by the gravitational force which is directed downwards, and is therefore during the entire throw given the constant acceleration $a = -g = -9.82\,\text{m/s}^2$ (we choose a negative *a* to get a positive direction upwards). The acceleration first lower the speed until the maximum height where the ball momentarily is at rest, after which its velocity changes direction and it begins to fall down to the ground with the same, free-fall acceleration. The choice of positive direction implies that the velocity is positive on the way up and negative on the way down.

b) The velocity changes its direction (sign) at the maximum height. Since the velocity function defined by $v = v_0 - gt$ is continuous, it must have the value 0 there.

Let t_{top} be the time when the ball reaches its maximum height.

$$v_0 - gt_{top} = 0 \Leftrightarrow t_{top} = \frac{v_0}{g}$$

Let t_{tot} be the time when the ball returns to the height of her hand. $t_{tot} > 0$. At this time, $s = 0$ again.

$$s = v_0 t_{tot} - \tfrac{1}{2} g t_{tot}^2 = 0 \Leftrightarrow t_{tot}\left(v_0 - \tfrac{1}{2} g t_{tot}\right) = 0 \Leftrightarrow v_0 - \tfrac{1}{2} g t_{tot} = 0 \Leftrightarrow$$
$$\Leftrightarrow t_{tot} = \frac{2v_0}{g}$$

We see that $\frac{t_{tot}}{t_{top}} = 2$ and hence that $t_{top} = 0.7\,\text{s}$.

c) We consider the motion from the hand to the maximum height. We can use the formula $s = v_0 t + \frac{1}{2} a t^2$ when studying such a constant acceleration if we know *the first* velocity v_0. Now, however, we only know *the last* velocity *v*, and therefore we have to derive a new formula. Since $v = v_0 + at$ we get

$$s = v_0 t + \tfrac{1}{2} a t^2 = (v - at)t + \tfrac{1}{2} a t^2 = vt - at^2 + \tfrac{1}{2} a t^2 = vt - \tfrac{1}{2} a t^2$$

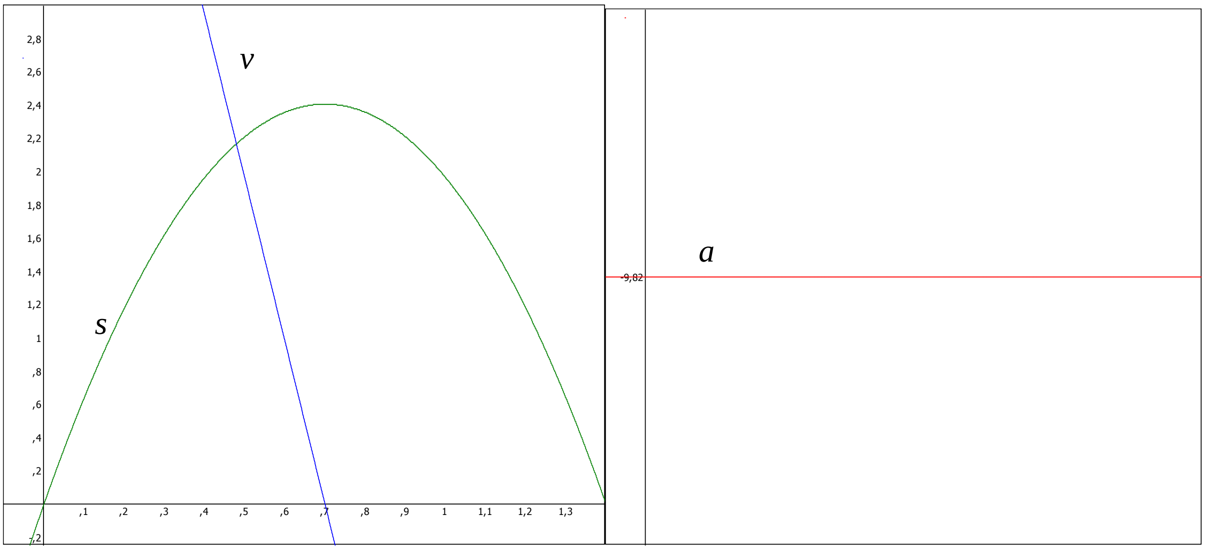
and as we know $v = 0\,\text{m/s}$, $t = t_{tot}/2$ and *a* we get $s \approx 2.4\,\text{m}$, which means that the ball's maximum height is 2.4 metres above the initial (hand) height.

d) Here we only consider the movement up to the maximum height. We get $v = v_0 + at \Leftrightarrow v_0 = v - at$ and hence $v_0 \approx 6.9\,\text{m/s}$.

e) We have already deduced that $v_{top} = 0\,\text{m/s}$.

f) Now we only consider the motion from the maximum height and back down to her hand. $v = v_0 + at \approx -6.9\,\text{m/s}$, which indeed was expected as this motion is "symmetrical" with respect to the time of the maximum height.

g) We have $s = v_0 t + \frac{1}{2}at^2$, $v = v_0 + at$ and $a = -9.82\,\text{m/s}^2$. The AlgoSim software is used to plot the graphs.



The formulas for *s*, *v* and *a* above as well as the corresponding graphs tell us that $v = \dfrac{ds}{dt}$ and

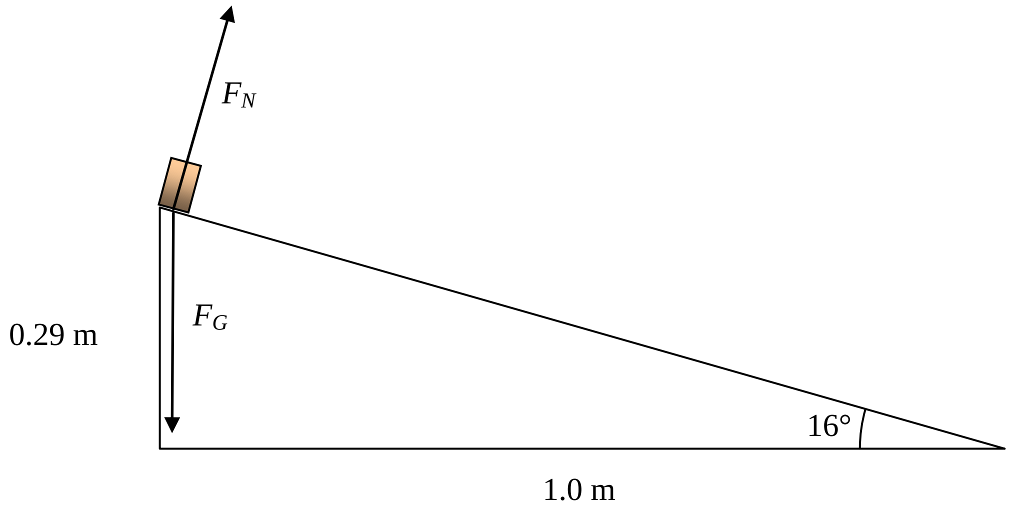
$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}.$$

Example 16

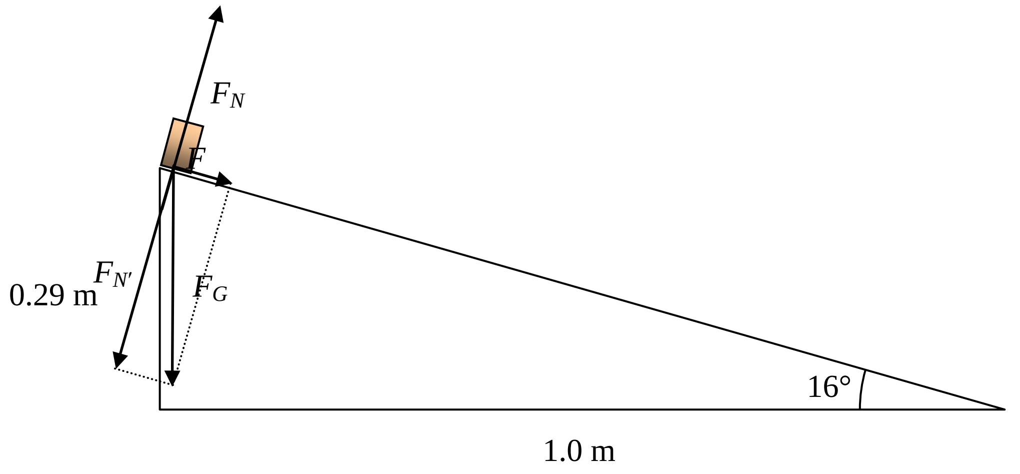
A block with a very smooth surface and mass $m = 0.5\,\text{kg}$ is placed at the top of an equally smooth surface of a sloped plane with a top angle of 16°, base 1.0 metres and height 0.29 metres. The block is then allowed to slide down the plane. Determine the speed of the block at the end of the plane. The block slides almost without friction, and we can therefore ignore the friction force.

Solution:

Now there are only two forces acting on the block: the gravitational force F_G downwards and a normal force F_N from the plane, holding the block up. (It would indeed be somewhat unexpected if the block fell right through the plane.) See the illustration.



We separate the F_G force into a component $F_{N'}$ parallel to the normal force F_N and another component F parallel to the upper surface of the place.



Since the block does not receive any acceleration in the direction parallel to the normal force F_N, we have $F_N = -F_{N'}$. We have no friction force, and hence the resulting force equals F. Using simple geometry, we can easily find that $F = F_G \sin 16° = mg \sin 16°$ and that the acceleration in the sliding direction will be $a = F/m = g \sin 16°$. The sliding time is given by $s = \frac{1}{2}at^2$ where the displacement $s = \sqrt{0{,}287^2 + 1}$ is given by Pythagoras' theorem. We have $t = \sqrt{2s/a}$ and the final velocity is given by $v = at$. With all figures inserted, we get $v \approx 2.4\,\text{m/s}$.

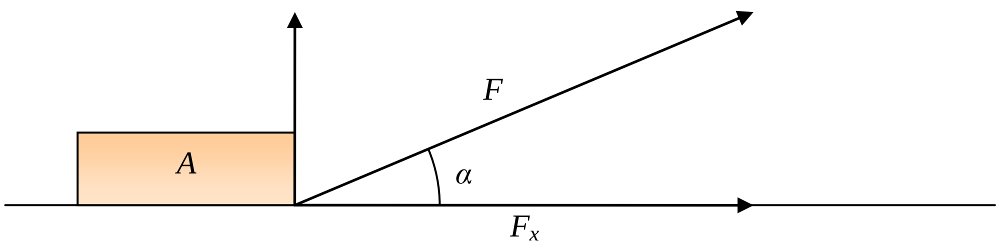
Answer: The final speed equals 2.4 m/s.

Energy

Energy is a very fundamental quantity in physics. An object is said to have energy, when it is able to perform a *work*, which can be considered as a transfer of energy or an "event". There are many different types of energy; everything in motion has kinetic energy, everything warmer than the absolute zero has thermal energy, a "flow" (current) of electrons has electric energy, light has radiation energy and carbohydrates has chemical energy, just to mention some examples. Energy can be transferred between different forms, but according to the *principle of energy* it is true that energy can *neither be created nor destroyed*. The chemical energy of the molecules of the gas can, for instance, be converted into kinetic energy of a car. Electric energy in a lamp can be converted into radiation energy (light) and in a radiator into thermal energy. Energy is designated E and is measured in the joule (J) unit.

Work

A work is a conversion of energy. Work is designated W and has the same unit as energy, i.e. J. If a force acts on an object at rest so that it begins to move, then the work equals the product of the force component in the direction of the velocity and the transported distance, by definition.



If the force F moves the object A the distance s, the performed work equals $W = F_x s = Fs\cos\alpha$. This equals the kinetic energy that A receives. By the equation $W = F_x s$ we see that the unit 1 J = 1 Nm = 1 kgm^2/s^2.

Mechanical energy

Kinetic energy

All objects in motion, i.e. objects with velocities $v \neq 0$, have *kinetic energy*. Now we want to derive an expression for the kinetic energy of a moving object, and in order to do so, we start with recalling that the kinetic energy equals the work resulting in the kinetic energy.

Let the object A be at rest and then displace it a distance s with the constant resulting force F, where $F \parallel s$, so that its new velocity becomes v. This takes the time t, and results in the work

$$W = F \cdot s = ma \cdot s = m \cdot \frac{v}{t} \cdot s = mv \cdot \frac{s}{t} = mv \cdot \bar{v} = mv \cdot \tfrac{1}{2} v = \tfrac{1}{2} mv^2 .$$

We realize that an object with mass m and velocity v has the kinetic energy

$$E_k = \tfrac{1}{2} mv^2 .$$

Potential energy

Potential energy can be thought of as “stored” work. If we perform a work *against* a force, the object will be loaded with potential energy. If then the force is allowed to act freely, this potential energy will be converted back into kinetic energy when the object is affected by the force. Since there are different types of forces, there are also different types of potential energy. We shall now study potential energy in the gravitational field, where the counteracting force is the gravitational force.

Imagine an object with mass m hold at rest in the air at height h. When released, the gravitational force drags the object down to the zero level (the ground) and the earth performs the work $W = Fs = mgh$. The object then receives kinetic energy and the surrounding air is heated. Just by being above the zero level, the object thus had the potential energy $E_p = mgh$. Please note that we here have chosen the ground level $h = 0\,\text{m}$ as a z*ero level* for the potential energy. We are always free to choose zero levels, as the magnitude of work depends on differences in potential energy, which obviously is independent of where the zero level is. (E_p is a linear function of h.)

The sum $E_k + E_p$ for an object is called the total *mechanical energy* E_m of the object.

Example 17

An object with mass m is hold 30 metres above the ground and is then released. At what velocity will it hit the ground, if we ignore air resistance?

Solution:

At the height $h = 30\,\text{m}$ the object has the potential energy $E_p = mgh$. When the object is released, it starts falling to the ground. During this fall E_p is constantly decreasing as h is decreasing. But energy can not disappear; instead the potential energy is converted into kinetic energy (the velocity is indeed increasing during the fall). At the ground level, where $h = 0\,\text{m}$ all potential energy has been converted into kinetic energy, and $\frac{1}{2}mv^2 = mgh$ where the left side equals the kinetic energy just before the object’s hitting the ground and the right side equals the initial potential energy. Solving the equation with respect to v gives $v = \sqrt{2gh}$. Note that the Galileo law is confirmed by this, since v is not a function of m. Numerically, we obtain $v \approx 24\,\text{m/s}$.

Answer: The object hits the ground with a velocity of 24 m/s.

Notice:

When the object hits the ground, it might bounce up again, i.e. the kinetic energy starts converting into potential energy again. Most often, however, a lot of mechanical energy is “lost” to thermal energy (“lost” because we most often neither need nor want the thermal energy). Eventually, the object lies at rest on the ground. At that moment, all mechanical energy has been converted into other forms, mostly thermal energy, as both v and $h = 0$.

If we had not ignored air resistance, mechanical energy had been converted into thermal energy during the fall, and the final speed v before hitting the ground would have been somewhat lower than what we just computed.

Example 18

Do example 15 using energy conservation formulas.

Solution:

The object is initially at rest at height $h = 0.29\,\text{m}$ with potential energy $E_m = E_p = mgh$. At the end of the plane $E_p = 0$ and thus $E_m = E_k$. $mgh = \frac{1}{2}mv^2$ and hence $v = \sqrt{2gh} \approx 2.4\,\text{m/s}$. We realise that the energy calculation is much simpler than the pure kinematic calculation.

Power

Power is designated *P* and is defined as work per unit of time.

$$\overline{P} = \frac{W}{t}$$

$$P = \lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t}$$

$$P = \frac{dE}{dt}$$

Power is measured in J/s = kgm^2/s^3 or W (watt). Do not confuse the work (*W*) quantity with the watt (W) power unit.

Example 19

Amanda can on 1 second lift a 3 kg bag up to a 1.4 metres high table, whereas Linux on 1.3 seconds can lift a 4.2 kg bag up to the same table. Whose work was most efficient (i.e. had the highest power)?

Solution:

Concerning Amanda we get $P_A = \frac{W_A}{t_A} = \frac{m_A gh}{t_A} \approx 41\,\text{W}$ whereas we concerning Linux get $P_L = \frac{W_L}{t_L} = \frac{m_L gh}{t_L} \approx 44\,\text{W}$. Thus, Linux lift was somewhat more efficient.

Example 20

A car with mass $m = 1500\,\text{kg}$ has the velocity $v_0 = 90\,\text{km/h}$. Then the engine performs a work $W = 0.23\,\text{MJ}$ on the car. We ignore friction and air resistance. At what velocity *v* does the car travel afterwards?

Solution:

The initial kinetic energy of the car was $E_{k0} = \frac{1}{2} m v_0^2$ where $v_0 = 90\text{ km/h} = 25\text{ m/s}$ and the new kinetic energy is $E_k = E_{k0} + W$. If we let $E_k = \frac{1}{2} m v^2$ and solve this equation for *v* we get $v = 30.52...\text{ m/s} \approx 110\text{ km/h}$.

Example 21

A light bulb has the effect $P = 75\text{ W}$ and is on for an hour. How much energy is then consumed?

Solution:

$P = 75\text{ J/s}$

$t = 3600\text{ s}$

$E = Pt = 0.27\text{ MJ}$

Extra material 1 – Electric current

An electric current is a flow of electrons. The voltage *U* is measured in the volt unit (1 V = 1 J/C) and states how much energy is released per unit charge when the electrons pass. Electric charge *Q* is measured in coulombs (C). An electron has the (negative) charge $e = 1.60 \times 10^{-19}\text{ C}$. The electrical current *I* states how great charge, i.e. in principal how many electrons, that passes per second and is measured in amperes (1 A = 1 C/s). Hence, the electrical power (energy per second) is given by $P = UI$.

Example 22

How many electrons pass per second in the lamp from example 21? Let us presume that the lamp is connected to a socket with a voltage of $U = 230\text{ V}$.

Solution:

$UI = P \Leftrightarrow I = P/U = 0.326...\text{ A}$

Let us call the number of passing electrons per second *n*. Then we have $ne = I$, and thus $n = I/e \approx 2.0 \times 10^{18}\text{ electrons/s}$.

Concerning an accelerating object, we can derive an alternative expression for the power.

$P = \frac{W}{t} = \frac{Fs}{t} = Fv$ where *v* is the speed of the object.

This formula explains why for instance cars have a top speed. The engine can product a maximal power *P*, which gives a traction force equal to $F = P/v$. When the velocity *v* increases, the *P*/*v* ratio and *F* decreases. Eventually, the traction force *F* becomes equal to the sum of the friction and air resistance forces $F_\mu + F_L$, acting backwards. Then F_{res} will be equal to zero, and no further acceleration is possible.

Friction and air resistance

This far, we have either not been interested in studying friction and air resistance or have simply ignored them, but we are now to study them more closely.

When a block is pulled on a horizontal surface, it will hook to the surface at a microscopic level, which will obstruct the motion. This force, which always acts backwards, is called the *friction force*, F_μ. The ratio between F_μ and F_N is called the *coefficient of friction* μ.

$$\mu = \frac{F_\mu}{F_N}$$

μ depends on the two surfaces in contact with each other. If a block is pulled forward with a constant force F on a horizontal surface and the block travels with a constant velocity, it must be true that $F_{res} = 0$ and thus $F = -F_\mu$, i.e. the friction force equals the negative of the traction force.

Air resistance obstructs a motion in air or any other gaseous substance. It has been discovered, that the air resistance F_L at "low" velocities approximately is proportional to the velocity and at "high" velocities approximately is proportional to the square of the velocity.

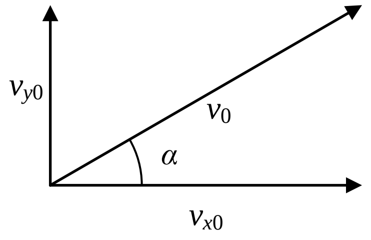
Thus we have that $F_L = kv^n$ where $n \in \{1,2\}$.

If, for instance, a parachutist jumps out of an airplane, the gravity F_G will at the beginning be grater than the air resistance F_L. Therefore, the resulting force will be grater than zero and pointing downwards, and the parachutist will accelerate downwards. Eventually, however, when the velocity has increased sufficiently, F_L will be equal to F_G, and then F_{res} and also the acceleration a will be equal to zero. Parachutists, and other objects falling in air, will hence reach a maximum velocity. (If an object is falling freely (i vacuum), however, no maximum speed will of course be reached.)

Curved motion

We are now to study curved motion. As an example of this, we shall study *projectile motion*, which occur when objects are given initial velocities in gravitational fields.

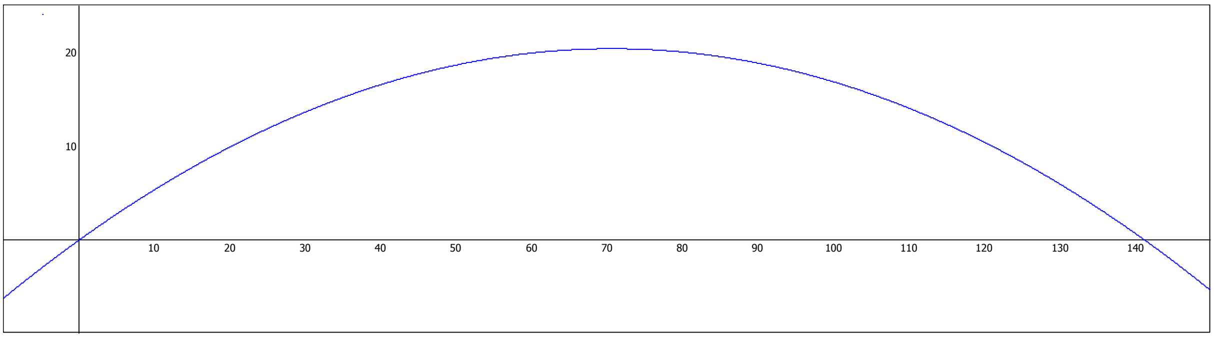
Let us imagine that we, standing on earth, throw away a ball with the velocity v_0 in an up-and-right direction. We are able to separate the velocity vector into a horizontal and a vertical component. We get $v_{x0} = v_0 \cos\alpha$ and $v_{y0} = v_0 \sin\alpha$ where α is the direction of the velocity.



Ignoring air resistance, no horizontal forces act on the ball, and hence $a_x = 0$ and the horizontal motion is uniform, i.e. $s_x = v_{x0}t$. The gravity of the ball, however, implies a downward acceleration $a_y = -g$, and therefore $s_y = v_{y0}t - \frac{1}{2}gt^2$.

$\begin{cases} s_x = v_{x0}t \\ s_y = v_{y0}t - \frac{1}{2}gt^2 \end{cases} \Leftrightarrow s_y = v_{y0}\dfrac{s_x}{v_{x0}} - \frac{1}{2}g\left(\dfrac{s_x}{v_{x0}}\right)^2$, and we are able to plot a graph $s_x \mapsto s_y$ depicting the motion.

If we, for instance, let $\alpha = 30°$ and $v = 40\,\text{m/s}$ we obtain the following graph.



We can very easily compute that the ball will land approximately 140 metres away and that the ball will be in the air for little more than four seconds. (Solve $s_y = 0$ with respect to *t*.)

Maximal length of projectile motion

What angle α results in the maximum length s_x of the projectile motion?

We know that the ball lands where $s_y = 0$.

$$s_y = v_{y0}t - \tfrac{1}{2}gt^2 = t\left(v_{y0} - \tfrac{1}{2}gt\right) = 0$$

$$t = 0 \text{ or } v_{y0} - \tfrac{1}{2}gt = 0$$

$$v_{y0} - \tfrac{1}{2} gt = 0 \Leftrightarrow t = \frac{2v_{y0}}{g}$$

The duration of the ball's being in the air is therefore $t = \dfrac{2v_{y0}}{g}$.

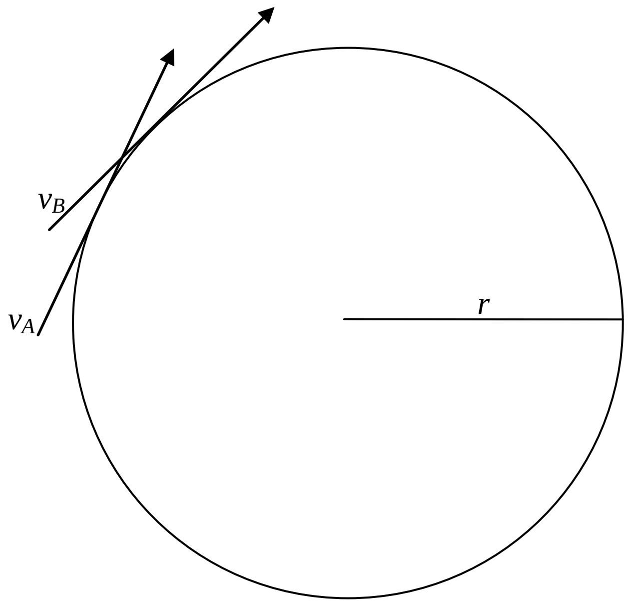
We know that $s_x = v_{x0}t = v\cos\alpha \cdot \dfrac{2v\sin\alpha}{g} = \dfrac{v^2}{g} \cdot 2\sin\alpha\cos\alpha = \dfrac{v^2}{g}\sin 2\alpha$

$$\frac{ds_x}{d\alpha} = \frac{2v^2}{g} \cdot \cos 2\alpha$$

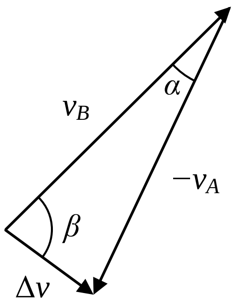
As $\dfrac{2v^2}{g} > 0$ we have $\dfrac{ds_x}{d\alpha} = 0$ when $\cos 2\alpha = 0$. Since $\alpha \in [0°, 90°]$ it is necessary that $2\alpha = 90° \Leftrightarrow \alpha = 45°$. We realize that this constitutes a maximum for s_x because $\dfrac{ds_x}{d\alpha}$ is positive when $\alpha < 45°$ and negative when $\alpha > 45°$. The maximal length of projectile motion is therefore attained when the initial velocity angle equals 45°, i.e. when the two components of the initial velocity are equal to each other.

Circular motion

An object moving with constant speed in a circular path is constantly accelerating as the velocity, and more precisely its direction, constantly is changing. We also realize this from Newton's second law, as the object would follow a linear path if no resulting force was acting on it. This acceleration is called the *centripetal acceleration*, and the resulting force implying this acceleration is called the *centripetal force*. The change of direction per unit of time, of course, gets greater for shorter radii and greater for higher speeds. The centripetal acceleration should therefore be a function of both the radius of the path and the speed of the particle. Now we want to find the magnitude and direction of the centripetal acceleration. In order to accomplish that, we are going to use the definition of acceleration, $a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$.

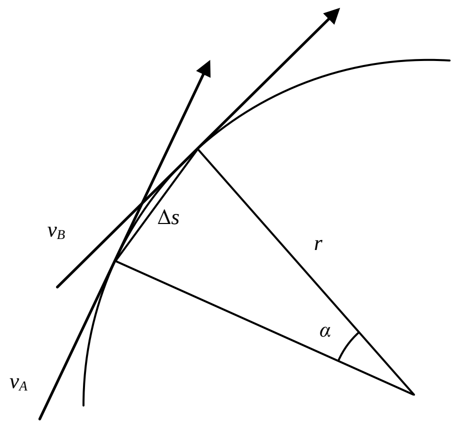


Let the velocity be v_A at time t_A and v_B at time t_B. Then $\overrightarrow{\Delta v} = \overrightarrow{v_B} - \overrightarrow{v_A}$.



α is the angle between the velocity vectors. As $|v_A| = |v_B|$, we will get an isosceles triangle. When $\Delta t \to 0$ the directions of the velocities approach each other, i.e. $\alpha \to 0$. The angular sum of the triangle gives that $\beta \to 90°$, i.e. $\Delta v \perp v_B$. As v_B is a tangent to the path, Δv must be a normal to it, and must be directed towards the centre of the circular path. Being a scalar product of Δv, the acceleration $a = \lim_{\Delta t \to 0}\left(\Delta v \cdot \frac{1}{\Delta t}\right)$ will get the same direction.

Let us magnify the part of the circle above that contains the two tangents v_A and v_B. Let us also draw Δs.



We obtain a new isosceles triangle which has the same angles as the one above. These triangles must therefore be similar to each other, and we have $\frac{\Delta v}{\Delta s} = \frac{v}{r}$.

$$\frac{\Delta v}{\Delta s} = \frac{v}{r} \Leftrightarrow \frac{\Delta v}{\Delta s}\frac{\Delta s}{\Delta t} = \frac{v}{r}\frac{\Delta s}{\Delta t} \Leftrightarrow \frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta s}{\Delta t}$$

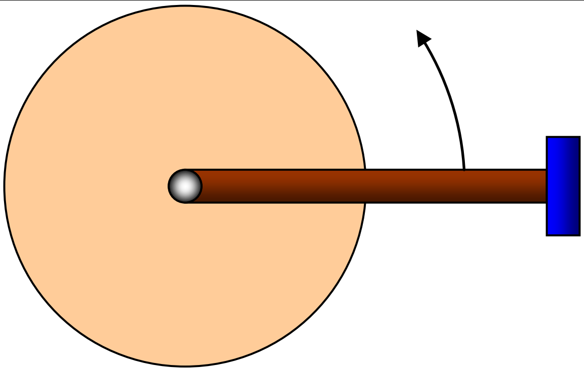
As $\frac{v}{r}\frac{\Delta s}{\Delta t} \rightarrow \frac{v}{r}v = \frac{v^2}{r}$ when $\Delta t \rightarrow 0$ we have in fact proven that $a = \frac{v^2}{r}$.

Let us summarize: If a body in motion follows a circular path at constant speed, then the centripetal acceleration is equal to $a = \frac{v^2}{r}$ and directed towards the centre of the path.

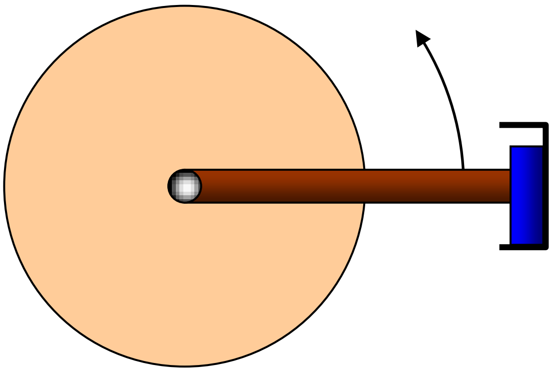
The force resulting in the centripetal acceleration is called the centripetal force and equals, according to Newton's second law, $F = m\frac{v^2}{r}$, where m is the mass of the body; the type of force resulting in the centripetal acceleration varies from case to case.

Example 22

Let us ignore gravitational forces. A blue box is fastened with glue on a rotating axis according to the illustration below. The centre of mass of the box follows a circular path with constant speed.



Electrical forces from the glue hold the box in its circular path, i.e. act as centripetal forces. If the box instead is hold by twines, then it is forces from the twines that keep the box to its path. If the box is lying in an inwards-turned container (see illustration below), then it is normal forces from the container acting as centripetal forces.



It is, of course, the resulting force on an object in a circular path that acts as centripetal force. If we do not ignore the gravitational force, then gravity will act in the same direction as the centripetal force when the box is at its top position, and the normal force will be smaller there. At its bottom position, the gravitational force acts in the opposite direction, and the normal forces will be greater there. If one was sitting in the container in the illustration above, one would feel "lighter" at the top of the path and "heavier" at the bottom of it. This connection between resulting force and normal force applies also when one is using an elevator. When the elevator is accelerating upwards, the resulting force and the gravitational force acts in opposite directions, the normal force will be grater and one will feel heavier. When the elevator is accelerating downwards, the resulting force and the gravitational force are acting in the same direction, the normal force will be lower and one will feel lighter.

Example 23

A car is running over a brow of a hill with constant speed. The motion can be approximated with a circular path with the radius 50 m. At which speed will the car just be about to lift from the ground when it is at the top of the brow?

Solution:

It is the gravitational force that acts as centripetal force on the car when it drives over the brow. Two forces are acting on the car: the normal force F_N from the road and the gravitational force F_G from the earth. The resulting force of these is the centripetal force F. We choose a positive direction downwards, and get $F_G = mg > 0$, $F > 0$ and $F_N \leq 0$.

$$F = F_G + F_N = m\frac{v^2}{r} \Leftrightarrow F_N = m\frac{v^2}{r} - mg$$

The fact that the car just is about to lift, means that no normal force from the ground is acting on it, i.e. that $F_N = 0$. We have $m\frac{v^2}{r} = mg \Leftrightarrow v = \sqrt{gr}$. We see that the speed is not a function of the mass of the car, and, which we had anticipated, that the speed gets higher for less obvious brows (larger radius) and higher gravitational field strengths. Will all values inserted we obtain $v \approx 22\,\text{m/s} \approx 80\,\text{km/h}$.

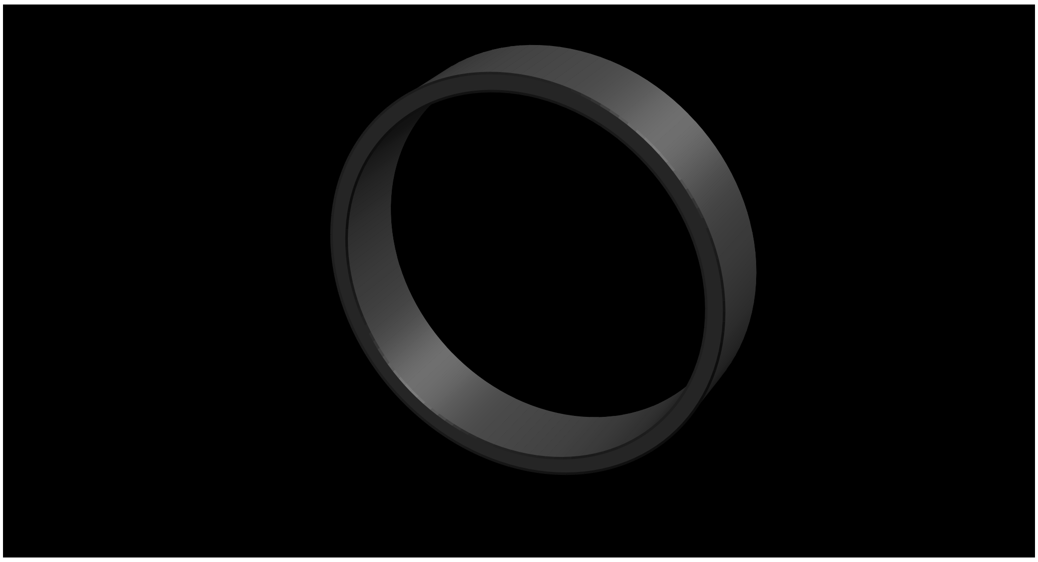
Answer: The car will just be about to lift from the ground when its speed equals 80 km/h. At higher speeds, the car will lift from the ground driving over the brow.

Notice:

If the car lifts from the ground it will no longer follow the circular path, and the formula $F = m\frac{v^2}{r}$ will no longer be useful for describing the motion of the car.

Example 24

Assume that we want to build a space station with a feeling of gravity. In a space station with "gravity" we would be able to move around more easily and our bodies would not be harmed as much as they would without the "gravity". The station will be build as a hollow cylinder rotating around its own axis. The floor of the station will be the inside of its outer envelope surface. (The outer) radius of the station is $r = 400\,\text{m}$. We wish to feel the same "gravitation feel strength" as on the earth's surface, i.e. 9.82 N/kg. At what velocity v must the station's outer surface then rotate, and at what angular velocity ω (radians/second) must it rotate?



Solution:

The fact that the experienced "gravitational field strength" on all bodies is to be 9.82 N/kg means that the centripetal force on all bodies at the station's outer surface must equal $F = 9.82m$ where *m* is the mass of the body. As $F = ma$ the centripetal acceleration must be $a = 9.82\ \text{m/s}^2$.

$$a = \frac{v^2}{r} \Leftrightarrow v = \sqrt{ar}$$

With all the values inserted we get $v \approx 62\ \text{m/s} \approx 223\ \text{km/h}$.

The angular velocity $\omega = \frac{\theta}{t} = \frac{2\pi}{O/v} = \frac{2\pi}{2\pi r/v} = \frac{v}{r} \approx 0.16\ \text{radians/s}$

Answer: The station must rotate around its own axis with an angular velocity of 0.16 radians/s.

Momentum

We have earlier defined the quantity energy as the product of force and displacement. The advantage of the quantity energy is that the sum of energy is preserved – "energy can neither be created nor destroyed". We are now to make another definition.

For a body with mass *m* and velocity *v* we define the *momentum p* as

$$p = mv .$$

The unit of momentum is hence kilogram metres per second (kgm/s). We see that momentum is a vector quantity with the same direction as the velocity. We are soon to see that also this quantity has mathematical advantages.

Impulse

If a force *F* acts on an object during time *t*, we say that the object receives an *impulse I*, defined by

$$I = Ft .$$

The quantity impulse gets the Newton seconds (Ns) unit and is a vector quantity with the same direction as the force. We shall see that there is a simple connection between the change of an object's momentum and the impulse acting on the object.

Let the object *A* have mass *m* and velocity v_1. A constant force *F* acts on the object during time *t*, which results in an acceleration *a* giving *A* the velocity v_2. Then the change in momentum $\Delta p = mv_2 - mv_1 = m(v_2 - v_1) = m \cdot \Delta v = mat = Ft = I$.

We see that an impulse on an object changes its momentum and that the impulse precisely equals the change of momentum, $I = \Delta p$.

We realize that the unit $1\,\text{kgm/s} = 1\,\text{Ns}$, which we also can realize directly from the definition of force: $1\,\text{N} = 1\,\text{kgm/s}^2$.

Preservation of momentum

Imagine two objects *A* and *B*. These have the masses m_A and m_B and velocities v_A and v_B, respectively. Their momentums are $p_A = m_A v_A$ and $p_B = m_B v_B$. Now imagine that they are hovering freely in a spaceship, and that they are on collision course with each other. During the collision, *A* acts on *B* with a force *F*, and according to Newton's third law, *B* acts on *A* with the force −*F*. The collision lasts during the time *t* and changes the velocities of *A* and *B* to u_A and u_B, respectively.

The impulse on *B* is $I_B = Ft$ and the impulse on *A* is $I_A = -Ft$. The total momentum before the collision was $p_{before} = m_A v_A + m_B v_B$, and the new momentum after the collision will be $p_{after} = m_A u_A + m_B u_B = (m_A v_A - Ft) + (m_B v_B + Ft) = m_A v_A + m_B v_B = p_{before}$.

We see that the momentum has been *preserved*. This result applies to all systems of objects, as long as no external (resulting) force acts on them. This is the advantage of calculating with momenta. We summarize:

If no external (resulting) force acts on an object, then the total momentum of the system (i.e. the vector sum of the momenta of all objects inside the system) is constant. (The law of preservation of momentum)

Example 25

Two stones are hovering freely inside a spaceship. They have the masses $m_A = 2\text{ kg}$ and $m_B = 1\text{ kg}$, respectively, and are approaching each other with the speeds $v_A = 2\text{ m/s}$ and $v_B = -3\text{ m/s}$, respectively, along the straight line between the stones. Their total momentum before the collision is $p_{before} = 1\text{ kgm/s}$. According the law of preservation of momentum, the total momentum after the collision will be $p_{after} = 1\text{ kgm/s}$. As the equation $p_{after} = m_A u_A + m_B u_B$ has two unknowns, i.e. the final velocities u_A and u_B, we are unable to determine those without more data about the collision.

If we, however, do know one of the velocities, we are able to calculate the other by using the preservation law. A special case is when the two objects *stick together* after the collision. Then $u_1 = u_2$ and we only have one unknown.

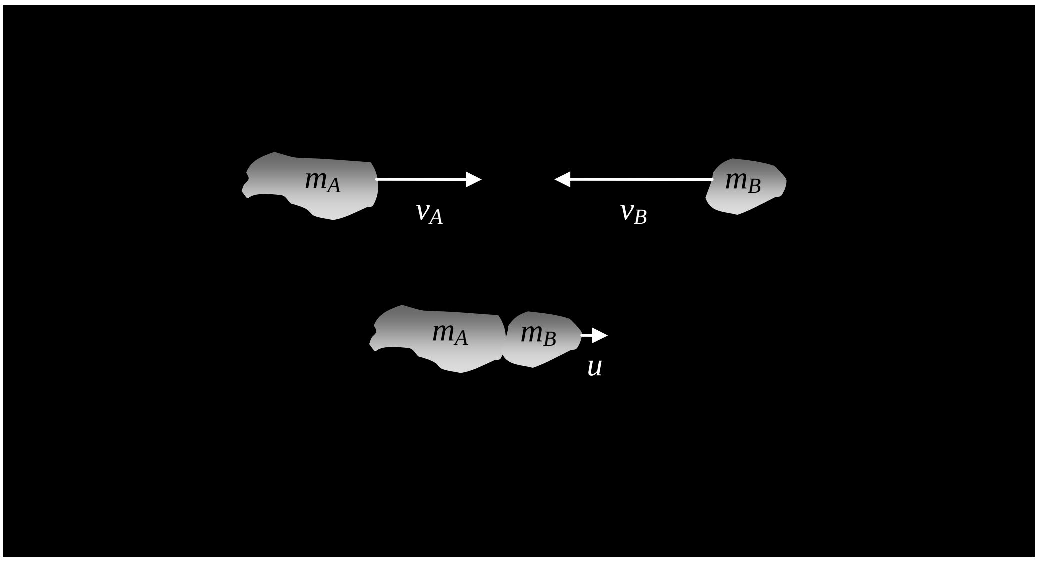
Example 26

The two stones from example 24 stick together after the collision. At what velocity do they hover afterwards?

Solution:

$p_{after} = m_1 u + m_2 u = 1\text{ kgm/s}$

$u = \frac{1}{3}\text{ m/s}$



Answer: The stones hover with the velocity $u = \frac{1}{3}\text{ m/s}$ in the direction that *A* had before the collision.

At a collision inside a system unaffected by external (resulting) forces, the momentum is preserved. The kinetic energy, however, might very well be altered. In the above example the kinetic energy drops from 8.5 J to 0.1666… J (we get thermal energy instead).

Special relativity

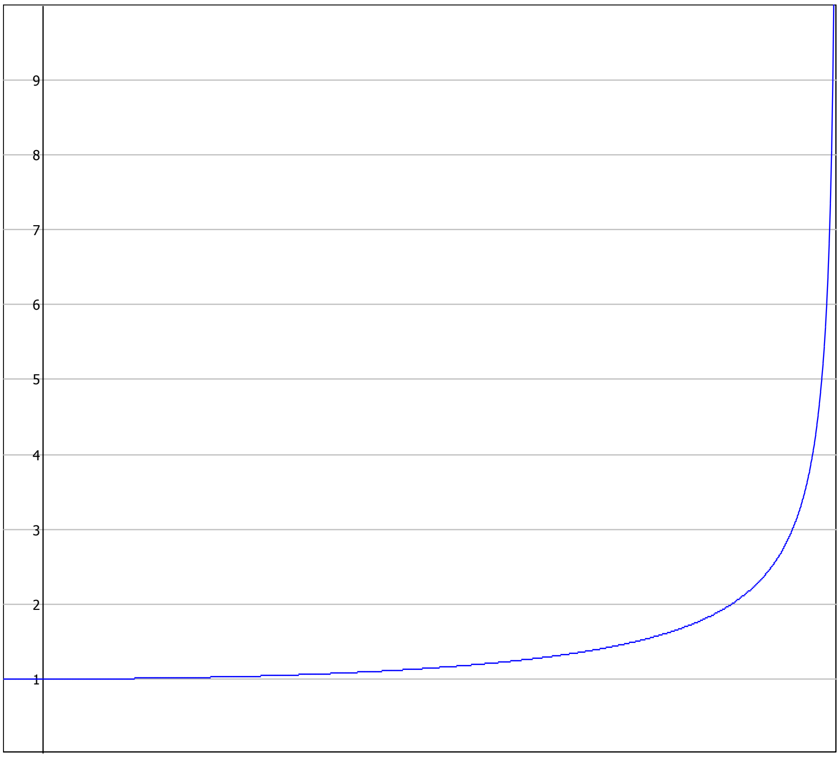
The mathematical models we have described in this document are in fact only approximations of more general (and somewhat more complicated) models. These approximations are however very accurate as long as the observed objects move at velocities much smaller than the speed of light, denoted c. $c = 299\,792\,458\ \text{m/s}$, which is a quite extraordinary speed.

One is quite easily able to deduce that time does not always go equally fast, if one considers some observations of electromagnetic radiation, i.e. light. Let A be an observer in a “vehicle” travelling at a speed v relative to another observer B outside the vehicle. A process that A observes as taking the time t_A, B observes as taking the time t_B. Intuitively, most people would probably say that $t_A = t_B$. That is not the case, however, as $t_B > t_A$. More precisely we have

$$t_B = \gamma t_A$$

where the *Lorentz factor* $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$. We realize that $\gamma \approx 1$ if $v << c$, i.e. in almost all every-day situations. Thus, in almost all every-day situations it is true, which satisfies intuition, that $t_A \approx t_B$.

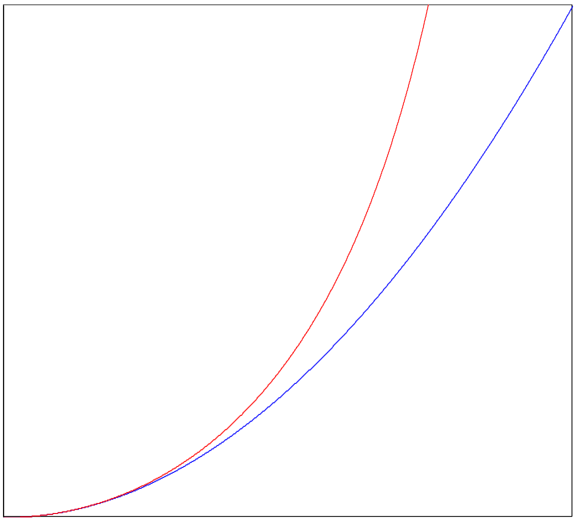
To get a sense of the function $v \mapsto \gamma$ for $v \in [0, c[$ we use AlgoSim to plot its graph.

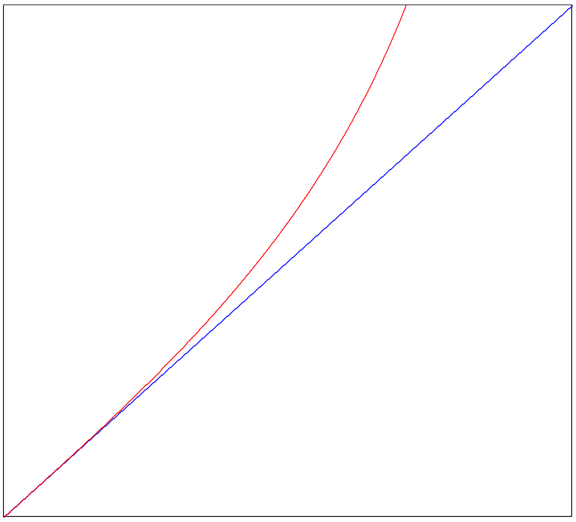


We see that the approximation’s errors in the classical mechanics in most applications should be insignificant even concerning (classically speaking) very high velocities v. Concerning velocities near c, however, we must use the more general, the so called *relativistic*, models. Without making any derivations, we will present a pair of common relativistic formulae.

- The kinetic energy $E_k = mc^2(\gamma - 1)$
- The momentum $p = \gamma mv$

If $v << c$ we have that $mc^2(\gamma - 1) \approx \frac{1}{2}mv^2$ and $\gamma mv \approx mv$, i.e. the classical approximations are valid. We can plot the functions $v \mapsto E_k$ and $v \mapsto p$ for $v \in [0, c[$, both with the classical and the relativistic expressions, in order to compare them. We set $m = 1\,\text{kg}$.





The values of the classical expressions are blue, whereas those of the relativistic expressions are red.

Example 27

Can an object travel with a speed greater than the speed of light?

Solution:

The kinetic energy is $E_k = mc^2(\gamma - 1)$, but $E_k \to \infty$ as $v \to c$. Thus, an infinite amount of work must be done in order to accelerate an object to the speed of light.

Experiments

We have previously considered a physician a seeker of truth. The goal with the mathematical models we have used is that they are going to reflect *reality*. And the best way to confirm this is to test them, to *experiment*. Many such experiments can very easily be done at home. As an example of such a test, we will take free fall.

Experiment 1 – Free fall

A weight with mass $m = 0.1\,\text{kg}$ is released from the height $h = 2.35\,\text{m}$ and is then allowed to fall freely in a room, except from air resistance, which should be quite moderate. We are, using our models, theoretically going to determine how long the fall should last, and then, experimentally, test how long the fall really lasts.

Theoretically we have $h = \frac{1}{2}gt^2 \Leftrightarrow t = \sqrt{2h/g} = 0.6918...\,\text{s}$.

The experiment was repeated 14 times, and the average fall time was calculated to 0.69 seconds. We have thus confirmed the theory.

Experiment 2 – Centripetal force and gravitational force

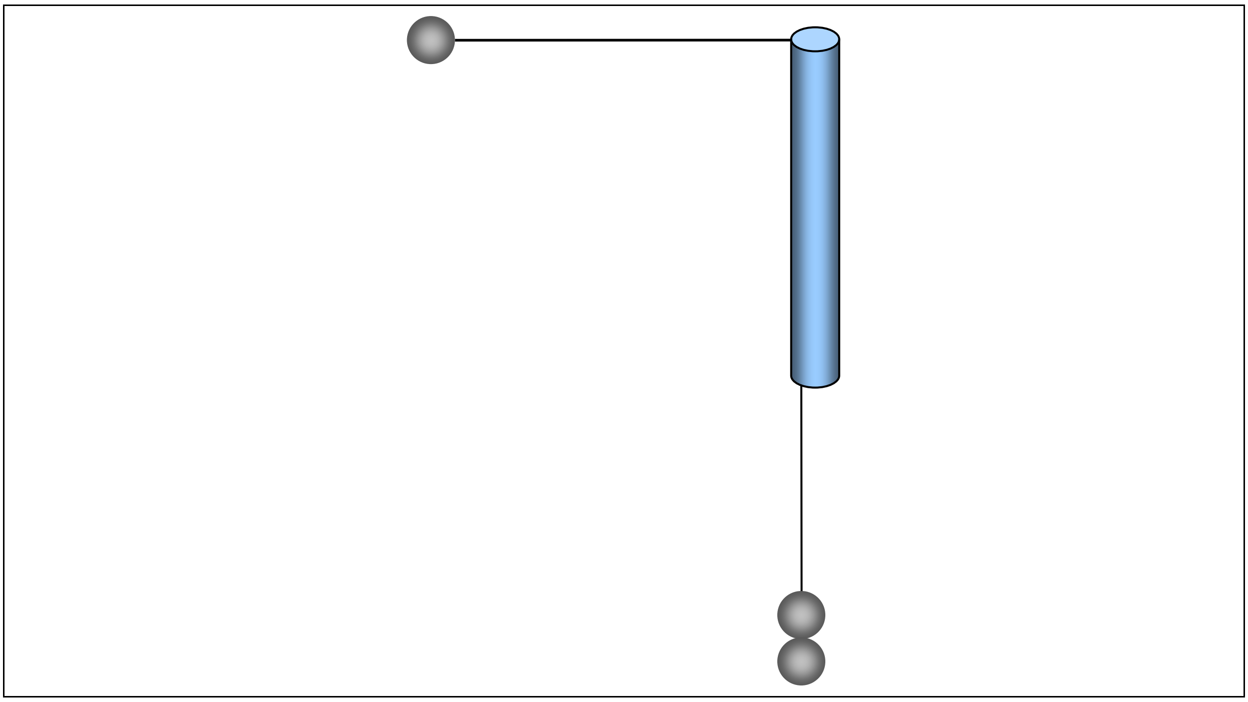
A heavier and a lighter weight are attached to the ends of a small string. The string goes through a pipe with smooth edges. The experimenter held the pipe in his hand and swung it over his head so that the lighter weight's motion described a circular path with constant speed. Speed and radius were adjusted so that the heavier weight, which hanged right down, was in equilibrium (i.e. had neither vertical velocity nor acceleration).

We wanted to determine the ratio between the masses of the two weights.

The lighter weight had a centripetal acceleration and hence also a centripetal force, equal to the weight of the heavier weight.

The centripetal force $F_c = m_1 a = m_1 \dfrac{v^2}{r} = m_1 \dfrac{(s/t)^2}{r} = m_1 \dfrac{(2r\pi/t)^2}{r} = \dfrac{4m_1 r\pi^2}{t^2}$ where $t = 0.9\,\text{s}$ and $r = 0.38\,\text{m}$ whereas the heavier weight's weight $F_G = m_2 g$. The equation $F_c = F_G$ is equivalent to $\dfrac{m_2}{m_1} = \dfrac{4r\pi^2}{t^2 g} \approx 1.9$.

In reality, the heavier weight consisted of two lighter weights put together, and thus the radio m_2/m_1 should have been approximately equal to 2. The measurement hence illustrates a relative error of 5.7 %, which should be considered acceptable considering the practical difficulties performing this experiment. We conclude that this confirms the theory.



Conclusion

In this document, we have presented a simple introduction to the models of classical mechanics describing the physical world. We have also studied the limitations of the models and compared them to more general ones, and have emphasised the importance of experimenting in order to confirm that the theories really are useful and really are describing nature.